Examples of infinite direct sums of spectral triples

Kevin Falk
Centre de Physique Théorique, 13288 Marseille, France
Aix Marseille Université, 13288 Marseille, France
Université de Toulon, 13288 Marseille, France
CNRS UMR 7332, 13288 Marseille, France

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A B S T R A C T
We study two ways of summing an infinite family of noncommutative spectral triples. First, we propose a definition of the integration of spectral triples and give an example using algebras of Toeplitz operators acting on weighted Bergman spaces over the unit ball of \( \mathbb{C}^n \). Secondly, we construct a spectral triple associated to a general polygonal self-similar set in \( \mathbb{C} \) using algebras of Toeplitz operators on Hardy spaces. In this case, we show that we can recover the Hausdorff dimension of the fractal set.

1. Introduction and motivation

The main idea of Connes’s noncommutative geometry is to characterize the geometry of a space in the language of algebras [11]. We know for instance that a compact Hausdorff space can be equivalently seen as the commutative C*-algebra of continuous functions living on it. By analogy, a noncommutative algebra would correspond to a space of quantum nature: a noncommutative space. More precisely, the algebraic description of a Riemannian manifold is based on the notion of unital spectral triple, consisting of the data \((A, H, D)\), where \(A\) is an involutive unital \(*\)-algebra faithfully represented on a Hilbert space \(H\) via a representation \(\pi\), and \(D\) is a selfadjoint operator acting on \(H\) with compact resolvent and such that for any \(a \in A\), \(\pi(a)\) maps \(\text{dom}(D)\) into itself, and \([D, \pi(a)]\) extends to a bounded operator on \(H\). When \(A\) is not unital, replace the compactness of the resolvent by the compactness of \(\pi(a)(D - \lambda)^{-1}\) for any \(a \in A\) and \(\lambda \notin \text{Spec}(D)\); the induced triple is then called nonunital. Among the various geometric entities which are encoded in the spectrum of \(D\), we are interested in the so-called spectral dimension, defined as the quantity

\[
d := \inf\{s \in \mathbb{R}, \text{Tr}[D]^{-s} < +\infty\}.
\]

As easily checked, the direct sum of a finite number of spectral triples is again a spectral triple. We are interested here in integrations of spectral triples which consist, roughly speaking, of the direct sum of an infinite number of spectral triples. Such constructions have already been encountered in [2]: the spectral triple related to the Berezin–Toeplitz quantization over a smoothly bounded strictly pseudoconvex domain of \(\mathbb{C}^n\) can be viewed as the integration of an infinite family of spectral triples based on algebras generated by Toeplitz operators acting on weighted Bergman spaces.

The first idea is the following: given a countable family of spectral triples \((A_m, H_m, D_m)_{m \in \mathbb{N}}\) (commutative or not), the corresponding infinite direct sum \(\bigoplus_{m \in \mathbb{N}}(A_m, H_m, D_m)\), might not be necessarily a spectral triple again. Indeed, as \(m\) tends to infinity, the boundedness of the representations of \(A_m\) is not preserved by \(\bigoplus_{m \in \mathbb{N}}(A_m, H_m, D_m)\).
\( \mathcal{D}_m \), or the compactness of the resolvent of the direct sum of all operators \( \mathcal{D}_m \) is hard to control in general and the sum may fail to converge. In order to control the behavior of the operators \( \mathcal{D}_m \), we multiply them by some coefficients \( \alpha_m \in \mathbb{R} \setminus \{0\} \).

Surprisingly, a strong link exists between direct summations of spectral triples and fractal sets, but before describing the second approach, let us recall some previous results on the topic. Since the works of A. Connes [1, Chapter 4, 3.e], we know that noncommutative geometry can detect the topology of fractal sets: it is shown that a commutative spectral triple involving \( C^* \)-algebra of continuous functions over the Cantor set can be used to recover its Hausdorff dimension and the Hausdorff measure. Later on, D. Guido and T. Isola proposed a commutative spectral triple, also based on a discrete approximation of the fractal, and extend Connes’ result to more general self-similar sets in \( \mathbb{R}^n \) [3, Chapter 7, [4] (the existence of such spectral triples was already conjectured in M. Lapidus’ paper [5]), and [6]. In [7, Sections 4 and 5], authors introduce Dirac operators and spectral triples related to Dirichlet forms over a locally compact separable metric space. This work sets up a valid framework for the spectral study of fractal sets. See also [8] for a review of open problems and questions about the links between analysis and spectral geometry on fractal sets.

In the latter works, each spectral triple is directly built over the fractal set. The approach we follow in the present paper is a constructive one: decompose the considered fractal set as the union of an infinite number of subdomains and associate to each of them a spectral triple. The spectral triple over the whole fractal set is obtained after the direct summation of all these spectral triples. This construction has already been used in [9–11] to recover the Hausdorff dimension and the metric on \( p \)-summable infinite trees and the Sierpinski gasket, and also in [12] to study the Hausdorff dimension of the Sierpinski gasket (and pyramid), its metric and describe its \( K \)-homology group.

For simplicity reasons, we restrict our study to self-similar sets \( E \) of the plane \( \mathbb{C} \) which can be expressed as

\[
E = E_0 \cup \bigcup_{k=1}^{N} F_k(E_0) \cup \bigcup_{k,l=1}^{N} F_k \circ F_l(E_0) \cup \ldots,
\]

where the overline means taking the closure, \( E_0 \) is a polygonal Jordan curve in the complex plane or the unit disk, and \( (F_k)_{k=1,...,N} \) is a finite family of contracting similarities.

The paper is organized as follows.

We present in Section 2 some sufficient conditions for the sum to be a spectral triple and we give an example of such integration using Toeplitz operators over the unit ball of \( C^* \).

We show in Section 3 that it is possible to build a noncommutative spectral triple over such sets, involving algebras of Toeplitz operators, and whose spectral description corresponds to the Hausdorff dimension of \( E \).

### 2. Abstract integration of spectral triples

#### 2.1. Conditions of integrability

**Lemma 2.1.** Let \( (\mathcal{H}_m)_{m \in \mathbb{N}} \) be a family of Hilbert spaces, \( (\mathcal{D}_m)_{m \in \mathbb{N}} \) be a family of unbounded selfadjoint operators with corresponding dense domains \( (\text{dom}(\mathcal{D}_m)) \subset \mathcal{H}_m \) and \( (\alpha_m)_{m \in \mathbb{N}} \in (\mathbb{R} \setminus \{0\})^\mathbb{N} \). Let \( \mathcal{D}^\oplus := \bigoplus_{m \in \mathbb{N}} \alpha_m \mathcal{D}_m \) with domain

\[
\text{dom}(\mathcal{D}^\oplus) := \left\{ \sum_{m=0}^{N} v_m \in \mathcal{H}^\oplus, N \in \mathbb{N}, v_m \in \text{dom}(\mathcal{D}_m) \right\}.
\]

Then \( \mathcal{D}^\oplus \) is essentially selfadjoint, with selfadjoint extension \( \mathcal{D}^\ominus \).

**Proof.** Let \( v^\ominus := \bigoplus_{m \in \mathbb{N}} v_m \in \mathcal{H}^\ominus \). For any \( m \in \mathbb{N} \), the operator \( \mathcal{D}_m \) is densely defined so there is a sequence \( (v_{mj})_{j \in \mathbb{N}} \) of elements in \( \text{dom}(\mathcal{D}_m) \) converging to \( v_m \) as \( j \to \infty \). Thus for any fixed \( (m, j) \in \mathbb{N}^2 \), there is \( M_{mj} \in \mathbb{N} \) such that \( \|v_m - v_{mj,M_{mj}}\|_{\mathcal{H}_m} < 2^{-j} \) for any \( k \in \mathbb{N} \). Define for any \( j \in \mathbb{N} \) the vector \( w_j^{(m)} := \bigoplus_{m=0}^{j} v_{mj,M_{mj}} \in \text{dom}(\mathcal{D}^\ominus) \).

For each \( j \in \mathbb{N} \), \( w_j^{(m)} \in \text{dom}(\mathcal{D}^\ominus) \) and

\[
\|v^\ominus - w_j^{(m)}\|_{\mathcal{H}^\ominus} = \sum_{m=0}^{j} \|v_m - v_{mj,M_{mj}}\|_{\mathcal{H}_m} + \sum_{m > j} \|v_m\|_{\mathcal{H}_m} < 2^{-j} + \sum_{m > j} \|v_m\|_{\mathcal{H}_m} \xrightarrow{j \to \infty} 0.
\]

Thus for any \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that \( \|v^\ominus - w_N^{(m)}\|_{\mathcal{H}^\ominus} < \varepsilon \), which shows that \( \mathcal{D}^\ominus \) is densely defined.

Using the same reasoning and the fact that for any \( m \in \mathbb{N} \), \( \text{Ran}(\alpha_m \mathcal{D}_m \pm i) = \mathcal{H}_m \) (since \( \alpha_m \mathcal{D}_m \) is selfadjoint), it can be shown that for any \( v^\ominus \in \mathcal{H}^\ominus \) and \( \varepsilon > 0 \), there is \( N \in \mathbb{N} \) and \( w_N^{(m)} \in \text{dom}(\mathcal{D}^\ominus) \) defined as above such that \( \|v^\ominus - \mathcal{D}^\ominus w_N^{(m)}\|_{\mathcal{H}^\ominus} < \varepsilon \), thus \( \text{Ran}(\mathcal{D}^\ominus \pm i) \) is dense in \( \mathcal{H}^\ominus \).

The operator \( \mathcal{D}^\ominus \) is also symmetric since for any \( v^\ominus := \sum_{m=0}^{N} v_m \) and \( v^{\ominus'} := \sum_{m=0}^{N} v_m' \) in \( \text{dom}(\mathcal{D}^\ominus) \),

\[
\langle v^\ominus, v'^\ominus \rangle_{\mathcal{H}^\ominus} = \sum_{m=0}^{N} \langle \alpha_m \mathcal{D}_m v_m, v'_m \rangle_{\mathcal{H}_m} = \sum_{m=0}^{N} \langle v_m, \alpha_m \mathcal{D}_m v'_m \rangle_{\mathcal{H}_m} = \langle v^\ominus, \mathcal{D}^\ominus v'^\ominus \rangle_{\mathcal{H}^\ominus},
\]

which shows that \( \mathcal{D}^\ominus \) is essentially selfadjoint (see [13, Chapter VIII.2, Corollary p.257]).
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