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Equilibrium states for impulsive semiflows ☆

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We consider impulsive semiflows defined on compact metric spaces and give sufficient conditions, both on the semiflows and the potential functions, for the existence and uniqueness of equilibrium states. We also generalize the classical notion of topological pressure to our setting of discontinuous semiflows and prove a variational principle.

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1. Introduction

Impulsive dynamical systems may be interpreted as suitable mathematical models of real world phenomena that display abrupt changes in their behavior, and are described by three objects: a continuous semiflow on a metric space X; a set $D \subset X$ where the flow experiments sudden perturbations; and an impulsive function $I : D \to X$ which determines the change on a trajectory each time it collides with the impulsive set D. See for instance reference [20], where one may find several examples of evolutive processes which are analyzed through differential equations with impulses.

Dynamical systems with impulse effects seem to be the most adequate mathematical models to describe real world phenomena that exhibit sudden changes in their states. For example, the theoretical characterizations of wormholes [27], also called Einstein–Rosen bridges, seem to fit the description of the traverse effects an impulsive function I acting on a set D induces on a semiflow, thereby possibly creating odd shortcuts in space–time [28]. While at present it appears unlikely that nature allows us to observe a wormhole, these hypothetical entities, with unusual and inherently unstable topological, geometrical and physical properties, show up as valid solutions of the Einstein field equations for the gravity. We also refer the reader to the reference [20], where other examples of nature evolution processes are analyzed within the new branch of differential equations with impulses; in addition, see [3,8,13,14,16,18,21,22,26,29].

For many years the achievements on the theory of impulsive dynamical systems concerned the behavior of trajectories, their limit sets and their stability; see e.g. [5,6,10,11,19] and references therein. The first results on the ergodic theory of impulsive dynamical systems were established in [1], where sufficient conditions for the existence of invariant probability measures on the Borel sets were given. Afterwards, it was natural to look for some special classes of invariant measures. So far, a useful approach has been to use potential functions and finding equilibrium states. However, as the classical notion of topological entropy requires continuity and impulsive semiflows exhibit discontinuities, it became necessary to introduce a generalized concept of topological entropy, and this has been done in [2]. Moreover, it was proved that the new notion coincides with the classical one for continuous semiflows, and also a partial variational principle for impulsive semiflows: the topological entropy coincides with the supremum of the metric entropies of time-one maps.

Our aim in this paper was to extend the results of [2] in two directions. Firstly we establish a variational principle for a wide class of potential functions; then we present sufficient conditions for the existence and uniqueness of equilibrium states for those potential functions. Once more, due to the discontinuities of the impulsive semiflows, we needed to define a generalized concept of topological pressure; and again we show that this new definition coincides with the classical one for continuous semiflows.

1.1. Impulsive semiflows

Consider a compact metric space (X, d), a continuous semiflow $\varphi : \mathbb{R}_0^+ \times X \to X$, a nonempty compact set $D \subset X$ and a continuous map $I : D \to X$ such that $I(D) \cap D = \emptyset$. Under these conditions we say that (X, φ, D, I) is an *impulsive dynamical system*. The first visit of each φ-trajectory to D will be registered by the function $\tau_1 : X \to [0, +\infty]$, defined as

$$\tau_1(x) = \begin{cases} \inf \{t > 0\colon \varphi_t(x) \in D\}, & \text{if } \varphi_t(x) \in D \text{ for some } t > 0; \\ +\infty, & \text{otherwise.} \end{cases}$$

The *impulsive trajectory* γ_x and the subsequent *impulsive times* $\tau_2(x), \tau_3(x), \dots$ (possibly finitely many) of a given point $x \in X$ are defined according to the following rules: for $0 \le t < \tau_1(x)$ we set $\gamma_x(t) = \varphi_t(x)$. Assuming that $\gamma_x(t)$ is defined for $t < \tau_n(x)$ for some $n \ge 1$, we set

$$\gamma_x(\tau_n(x)) = I(\varphi_{\tau_n(x)-\tau_{n-1}(x)}(\gamma_x(\tau_{n-1}(x)))).$$