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Extinction and stationary distribution of an impulsive stochastic chemostat model with nonlinear perturbation

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ABSTRACT

This paper investigates a new impulsive stochastic chemostat model with nonlinear perturbation in a polluted environment. We present the analysis and the criteria of the extinction of the microorganisms, and establish sufficient conditions for the existence of a unique ergodic stationary distribution of the model via Lyapunov functions method. The results show that both stochastic noise and impulsive toxicant input have great effects on the survival and extinction of the microorganisms. Moreover, we provide a series of numerical simulations to illustrate the analytical results.

1. Introduction

The chemostat is a device for continuous culture of microorganisms in laboratory, many scholars have done a lot of work on the dynamics modeling and analysis of the chemostat, and many good results were obtained [1–14]. Moreover, impulsive stochastic differential equations [15–19] are used more and more widely, the biological models with impulsive effects were studied by a lot of scholars [20–25].

By learning from previous researches, we find that the saturated growth rate may be more suitable than bilinear growth rate for many cases, see [26,27]. And the toxicant in the air pollution and water pollution environment is a threat to the survival of the exposed microorganisms. Consequently, it is important to discuss impulsive chemostat model with saturated growth rate in a polluted environment [28–32]. In [1], the author investigated a deterministic impulsive chemostat model with saturated growth rate in a polluted environment, this model is described by the following impulsive differential equation

$$\begin{cases} \left.\begin{aligned} dS(t) &= \left(Q(S_0 - S(t)) - \frac{\mu S(t)x(t)}{\delta(a + x(t))}\right)dt, \\ dx(t) &= \left(\frac{\mu S(t)x(t)}{(a + x(t))} - Qx(t) - rC_0(t)x(t)\right)dt, \\ dC_0(t) &= (kC_e(t) - gC_0(t) - mC_0(t))dt, \\ dC_e(t) &= -hC_e(t)dt, \end{aligned}\right\} t \neq n\tau, n \in \mathbb{Z}^+, \\ \Delta S(t) = 0, \Delta x(t) = 0, \Delta C_0(t) = 0, \Delta C_e(t) = u, t = n\tau, n \in Z^+, \end{cases} \quad (1)$$

where $S(t)$ represents the concentration of the unconsumed nutrient at time t, $x(t)$ represents the biomass of the population of microorganism at time t, $C_0(t)$ and $C_e(t)$ denote the concentrations of the toxicant in the organism and in the environment at time t. Q is the washout rate, S_0 is the concentration of the growth-limiting nutrient, μ is the maximal growth rate, δ is the yield of the microorganism $x(t)$ per unit mass of substrate, a is the half-saturation constant with units of concentration, r is the rate of decrease of the intrinsic growth rate, k represents environmental toxicant uptake rate per unit mass organism, g and m are organismal net ingestion and depuration rates of toxicant, respectively, h denotes the loss rate of toxicant from the environment itself by volatilization, u is the amount of pulsed input concentration of the toxicant at each τ. And all the parameters are positive.

As we know, the parameters of a system can be affected by environmental noises [33–44]. Therefore, it is necessary to consider the effect of environmental noises. In this paper, we assume that fluctuations in the environment will manifest themselves mainly as fluctuations in the flow rate of the chemostat. Moreover, we use

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nonlinear perturbation [45,46] for the flow rate of the chemostat. Then, an impulsive stochastic chemostat model with nonlinear perturbation in a polluted environment takes the following form

$$\begin{cases} \left.\begin{aligned} dS(t) &= \left(Q(S_0 - S(t)) - \frac{\mu S(t)x(t)}{\delta(a+x(t))}\right)dt + S(t)(\sigma_{11}+\sigma_{12}S(t))dW_1(t), \\ dx(t) &= \left(\frac{\mu S(t)x(t)}{(a+x(t))} - Qx(t) - rC_0(t)x(t)\right)dt + x(t)(\sigma_{21}+\sigma_{22}x(t))dW_2(t), \\ dC_0(t) &= (kC_e(t) - gC_0(t) - mC_0(t))dt, \\ dC_e(t) &= -hC_e(t)dt, \end{aligned}\right\} \\ \Delta S(t) = 0,\ \Delta x(t) = 0,\ \Delta C_0(t) = 0,\ \Delta C_e(t) = u,\ t = n\tau,\ n \in Z^+. \\ \times\, t \neq n\tau, n \in \mathbb{Z}^+, \end{cases} \tag{2}$$

For convenience, we first consider the following subsystem of system (2)

$$\begin{cases} \left.\begin{aligned} dC_0(t) &= (kC_e(t) - gC_0(t) - mC_0(t))dt, \\ dC_e(t) &= -hC_e(t)dt, \end{aligned}\right\} t \neq n\tau, n \in \mathbb{Z}^+, \\ \Delta C_0(t) = 0,\ \Delta C_e(t) = u,\ t = n\tau,\ n \in Z^+. \end{cases} \tag{3}$$

by Lemmas 2.1 and 2.2 of [1], we have

$$\lim_{t\to+\infty} \langle C_0(t)\rangle = \frac{ku}{h(g+m)\tau} \triangleq \bar{C}_0.$$

The paper is organized as follows. In Section 2, we obtain some of the main results. First, we explore the conditions for the extinction of the microorganisms. Then, we establish sufficient conditions for the existence of an ergodic stationary distribution. In Section 3, we provide a series of numerical simulations to illustrate the analytical results.

2. Main results

First, we give some notations, assumptions and some lemmas which will be used for our main results. Throughout this paper, unless otherwise specified, let $(\Omega, \mathscr{F}, \mathscr{F}_{t\geq 0}, P)$ stand for a complete probability space with a filtration $\mathscr{F}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous where $\mathscr{F}_0$ contains all P-null sets). Define $f^l = \inf_{t\in R_+} f(t)$, $f^u = \sup_{t\in R_+} f(t)$, here $f(t)$ is a bounded function on $[0, \infty)$, $\langle f(t)\rangle = \frac{1}{t}\int_0^t f(t)ds$, where $f(t)$ is an integrable function on $[0, \infty)$.

Assumption 2.1 [47]. There exists a bounded domain $U \subset E_d$ with regular boundary, then

(A1) In the open domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero;

(A2) If $x \in E_d \setminus U$, the mean time τ at which a path issuing from x reaches the set U is finite, and $\sup_{x\in K} E_x\tau < \infty$ for every compact subset $K \subset E_d$.

Remark 2.1. To validate (A1), we need to prove that there exists a positive constant $M > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq M\,|\,\xi\,|^2, x \in U, \xi \in E_d.$$

To validate (A2), we need to prove that there exists a non-negative C^2-function V and a neighborhood U such that, LV is negative for any $E_d \setminus U$.

Assumption 2.1 is a general assumption which is the condition for the following Lemma 2.1.

Lemma 2.1. *By Theorem 4.1 of [47], if Assumption 2.1 holds, the Markov process $X(t)$ has a stationary distribution $\mu(\cdot)$. And*

$$\mathbb{P}\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T f(x(t))dt = \int_{E_d} f(x)\mu(dx)\right\} = 1,$$

where f is an integrable function with respect to the measure μ. The proof is given in [47].

2.1. Extinction

In this section, we explore the condition for the extinction of the microorganism, which implies microculture failed.

Define

$$R_0^* = \frac{\frac{\mu}{a}\int_0^\infty x\pi(x)dx}{Q + r\bar{C}_0 + \frac{\sigma_{21}^2}{2}},$$

where for $x \in (0, +\infty)$

$$\pi(x) = Cx^{-2-\frac{2(2QS_0\sigma_{12}+Q\sigma_{11})}{\sigma_{11}^3}}(\sigma_{11}+\sigma_{12}x)^{-2+\frac{2(2QS_0\sigma_{12}+Q\sigma_{11})}{\sigma_{11}^3}} \times e^{-\frac{2}{\sigma_{11}(\sigma_{11}+\sigma_{12}x)}\left(\frac{QS_0}{x}+\frac{2QS_0\sigma_{12}+Q\sigma_{11}}{\sigma_{11}}\right)}, \tag{4}$$

here, C is a constant such that $\int_0^\infty \pi(x)dx = 1$.

Theorem 2.1. *Assume $R_0^* < 1$, let $(S(t), I(t), C_0(t), C_e(t))$ be the solution of system (2) with any initial value $(S(0), I(0), C_0(0), C_e(0^+)) \in \mathbb{R}_+^4$. Then*

$$\lim_{t\to+\infty} I(t) = 0 \ a.s.$$

Proof. First, we construct the following auxiliary equation

$$dX(t) = [QS_0 - QX(t)]dt + X(t)(\sigma_{11}+\sigma_{12}X(t))dB_1(t), \tag{5}$$

with the initial value $X(0) = S(0) > 0$.

Let $X(t)$ be the solution of Eq. (5), then using the comparison theorem for stochastic differential equation, we get

$$S(t) \leq X(t) \ a.s.$$

Setting

$$a(x) = QS_0 - Qx, \sigma(x) = x(\sigma_{11}+\sigma_{12}x), x \in (0, +\infty),$$

then we compute the following indefinite integral

$$\begin{aligned} \int \frac{a(t)}{\sigma^2(t)}dt &= \int \frac{QS_0 - Qt}{t^2(\sigma_{11}+\sigma_{12}t)^2}dt \\ &= \int \left(\frac{QS_0}{t^2(\sigma_{11}+\sigma_{12}t)^2} - \frac{Q}{t(\sigma_{11}+\sigma_{12}t)^2}\right)dt \\ &= \frac{2QS_0\sigma_{12}}{\sigma_{11}^3}\ln\frac{\sigma_{11}+\sigma_{12}t}{t} - \frac{QS_0}{\sigma_{11}t(\sigma_{11}+\sigma_{12}t)} \\ &\quad - \frac{2QS_0\sigma_{12}}{\sigma_{11}^2(\sigma_{11}+\sigma_{12}t)} - \frac{Q}{\sigma_{11}(\sigma_{11}+\sigma_{12}t)} \\ &\quad + \frac{Q}{\sigma_{11}^2}\ln\frac{\sigma_{11}+\sigma_{12}t}{t} + C \\ &= \frac{2QS_0\sigma_{12}+Q\sigma_{11}}{\sigma_{11}^3}\ln\frac{\sigma_{11}+\sigma_{12}t}{t} - \frac{QS_0}{\sigma_{11}t(\sigma_{11}+\sigma_{12}t)} \\ &\quad - \frac{2QS_0\sigma_{12}+Q\sigma_{11}}{\sigma_{11}^2(\sigma_{11}+\sigma_{12}t)} + C. \end{aligned}$$

Hence

$$e^{\int\frac{a(t)}{\sigma^2(t)}dt} = e^C\left(\frac{\sigma_{11}+\sigma_{12}t}{t}\right)^{\frac{2QS_0\sigma_{12}+Q\sigma_{11}}{\sigma_{11}^3}} e^{-\frac{1}{\sigma_{11}(\sigma_{11}+\sigma_{12}t)}\left(\frac{QS_0}{t}+\frac{2QS_0\sigma_{12}+Q\sigma_{11}}{\sigma_{11}}\right)}.$$

Next, we have