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Game-theoretic modeling of players' ambiguities on external factors

Jian Yang

Department of Management Science and Information Systems, Business School, Rutgers University, Newark, NJ 07102, United States

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ABSTRACT

We propose a game-theoretic framework that incorporates both incomplete information and general ambiguity attitudes on factors external to all players. Our starting point is players' preferences on return-distribution vectors, essentially mappings from states of the world to distributions of returns to be received by players. There are two ways in which equilibria for this preference game can be defined. Also, when the preferences possess ever more features, we can gradually add more structures to the game. These include real-valued functions over return-distribution vectors, sets of probabilistic priors over states of the world, and eventually the traditional expected-utility framework involving one single prior. We establish equilibrium existence results, show the upper hemi-continuity of equilibrium sets over changing ambiguity attitudes, and uncover relations between the two versions of equilibria. Some attention is paid to the enterprising game, in which players exhibit ambiguity-seeking attitudes while betting optimistically on the favorable resolution of ambiguities. The two solution concepts are unified at this game's pure equilibria, whose existence is guaranteed when strategic complementarities are present. The current framework can be applied to settings like auctions involving ambiguity on competitors' assessments of item worths.

1. Introduction

In the traditional expected-utility approach to games involving incomplete information, such as Harsanyi's (1967–1968), a player n's type t_n indicates that his state of the world ω comes from the subset Ω_{n,t_n} of the state space Ω. After observing his own type t_n, player n can form a probabilistic understanding $p_{n,t_n} \equiv (p_{n,t_n|t_{-n}})_{t_{-n}\in T_{-n}}$ about other players' types $t_{-n} \in T_{-n}$. Presented with others' strategies, he will strive to maximize his own expected utility, where expectation is taken with the aforementioned assessment p_{n,t_n}. Ellsberg (1961), however, argued that decision makers (DMs) often do not know the probabilities to be assigned to different states of the world. For instance, there are probably not enough data to estimate the chance of a new financial crisis to occur within the next two years; also, there has no precedent to be relied on to assess probabilities concerning the climate change due to human activities. Hence, to many situations the single-prior assumption on uncertain factors can be ill suited.

Starting from Schmeidler (1989), researchers resorted to non-conventional tools to help with single-agent decision making involving general ambiguity attitudes; see, e.g., Gilboa and Marinacci (2013). In a strategic setting involving incomplete information, failure to account for players' diverse ambiguity attitudes could lead to weird predictions or dangerous prescriptions. In auctions, especially those involving works of art, offshore oilfields, or electromagnetic spectra, participants often do not know for sure the actual worths to themselves of the item being auctioned; very likely, they are also uncertain about the distributions their competitors assign to the item's worths; in addition, some may fear losing the object more than they regret about overpaying for it. How can a model capture these features then?

We make an attempt to partially answer this question by pursuing a framework that allows players' diverse ambiguity attitudes to be considered along with strategic interactions involving incomplete information. It revolves around the concept of "*return-distribution vector*"s. In an n-player game, suppose all players adopt potentially random actions based on their types. Then, right after receiving his own type t_n but before knowing anything about others' types t_{-n} let alone the true state ω, player n should anticipate one return distribution say $\pi(\omega)$ per state $\omega \in \Omega_{n,t_n}$. He will certainly want to make the vector $\pi \equiv (\pi(\omega)|\omega \in \Omega_{n,t_n})$ as likable to himself as possible. A natural apparatus to express the "(n, t_n)"-player's taste is a strict preference relationship $\succ_{n,t_n}$ on all return-distribution vectors.

For example, it might be that $\Omega_{n,t_n} = \{\text{hot day, cold day}\}$ and the (n, t_n)-player's return space $R_{n,t_n} = \{\text{ice cream, chili soup}\}$. Then, one $\succ_{n,t_n}$ might dictate that "ice cream when it is hot and chili soup when it is cold" is strictly preferred to "either snack with a 50% chance on either type of a day", which is in turn strictly preferred to "chili soup when it is hot and ice cream when it is cold". The traditional expected-utility approach basically uses what we shall call a real-valued satisfaction function s_{n,t_n} on return-distribution vectors $\pi \equiv \left(\pi(\omega)|\omega \in \Omega_{n,t_n}\right)$ to facilitate

E-mail address: jyang@business.rutgers.edu.

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each (n, t_n)-player's preference relation:

$$\pi \succ_{n,t_n} \pi' \quad \text{if and only if} \quad s_{n,t_n}(\pi) > s_{n,t_n}(\pi'); \tag{1}$$

moreover, s_{n,t_n} is specially built from one single probabilistic prior ρ_{n,t_n} on the state space Ω_{n,t_n} and a utility function u_{n,t_n} on returns, in the fashion of

$$s_{n,t_n}(\pi) = s^0_{n,t_n}(\pi, \rho_{n,t_n}) = \int_{\Omega_{n,t_n}} \left[\int_{R_{n,t_n}} u_{n,t_n}(r) \cdot [\pi(\omega)](dr) \right] \cdot \rho_{n,t_n}(d\omega). \tag{2}$$

Our departure point is that, even without the s_{n,t_n}'s that help to substantiate the preference relations $\succ_{n,t_n}$ through (1), we can already define our preference game. The emphasis here, though, is not the consideration of preferences itself. In various strategic settings, this has been done by, e.g., Schmeidler (1969), Mas-Colell (1974), Shafer and Sonnenschein (1975), Khan and Sun (1990), and Grant et al. (2016). It is preferences on *return-distribution vectors* that we want to stress. We believe such preferences provide more flexibility than those on actions, action distributions, integrated return distributions, or return vectors.

For our preference game revolving around the relations $\succ_{n,t_n}$ that act on (n, t_n)-players' return-distribution vectors, we can recognize two equilibrium notions. The first, *action*-based interpretation leaves every player in control of his action whilst maintaining a long-term commitment to his portion of a behavioral equilibrium. The second, *distribution*-based interpretation ties every player's action to the outcome of a random device in a fashion consistent to his portion of an equilibrium. Under mild conditions, we show that action-based equilibria always exist. When the preferences $\succ_{n,t_n}$ connote ambiguity aversion, distribution-based ones will come into being as well; see Theorem 1. Both sets of equilibria are upper hemi-continuous in players' ambiguity attitudes; see Theorem 2. When the preferences are representable by real-valued functions s_{n,t_n} satisfying (1), our game is specialized to the so-called satisfaction kind. For this game, action-based equilibria will exist in general and so will distribution-based equilibria when the s_{n,t_n}'s are quasi-concave.

Under axioms associated with ambiguity aversion, Gilboa and Schmeidler (1989) legitimized the worst-prior form to be taken by a DM. In this form,

$$s_{n,t_n}(\pi) = \inf_{\rho \in P_{n,t_n}} s^0_{n,t_n}(\pi, \rho), \tag{3}$$

where P_{n,t_n} is a set of prior distributions on Ω_{n,t_n} and s^0_{n,t_n} is defined in (2). We call the special satisfaction game satisfying (3) an alarmists' game. Due to concavity of the s_{n,t_n}'s, it has both action- and distribution-based equilibria. We also step into the ambiguity-seeking territory that has not been well traversed since Ellsberg's (1961) pioneering work. Indeed, experiments involving human subjects showed that ambiguity-seeking traits could be equally prevalent; see, e.g., Curley and Yates (1989) and Charness et al. (2013). We also believe that optimistic assessments of uncertain gains drive people to participate in auctions, embark on exploratory journeys, and start new firms. Thus, the case opposite to (3) is equally if not more important. We call the corresponding game "enterprising" because

$$s_{n,t_n}(\pi) = \sup_{\rho \in P_{n,t_n}} s^0_{n,t_n}(\pi, \rho), \tag{4}$$

so that players make optimistic bets on favorable resolutions of their ambiguities.

For the preference game, rudimentary understandings on relations between the action- and distribution-based equilibria can be formed. Our message will become considerably sharper for the satisfaction game. For it, we can conclude that distribution-based equilibria will be action-based ones when players are ambiguity-seeking and the two types will be identical when players are ambiguity-neutral; see Theorem 3. Consequently, the distinction between the two versions of equilibria will cease to matter for the traditional expected-utility game; see Theorem 4. Our derivation relies on concepts like continuous kernels and their integrations, as well as intermediate results like Lemma 1 that might be of some value on its own. When we focus on pure equilibria, we show that any pure distribution-based equilibrium must also be a pure action-based one; see Theorem 5.

As the enterprising game is a special satisfaction game with convex s_{n,t_n} functions, any of its distribution-based equilibria is necessarily an action-based one. When confined to pure strategies, we also have the equivalence between the game's two types of equilibria; see Theorem 6. One technical result involved in its proof is Lemma 2. It is an extension of a well known finite-dimensional property, stating that the maximum of a convex function over a convex region in $\Re^d$ can always be achieved at extreme points. When equipped with strategic complementarity features, a special enterprising game can be shown to possess not only pure equilibria, but also those having monotone trends with respect to players' types as well as external conditions; see Theorems 7 and 8. These results can be considered as extensions of those achieved for the traditional counterpart by van Zandt and Vives (2007).

We go over literature in Section 2 and give our formulation in Section 3. The existence and continuity of the two types of equilibria are derived in Section 4. We next delve into various special cases in Section 5, and establish relationships between the two equilibrium concepts in Section 6. The special enterprising game is treated in Section 7. We discuss our framework's suitability to auctions in Section 8 and conclude the paper in Section 9.

2. Literature survey

Normal-form games incorporating general ambiguity attitudes have been studied. Dow and Werlang (1994) used convex capacities to model players' beliefs about opponents' behaviors and arrived at equilibrium belief profiles. Eichberger and Kelsey (2000) extended the study to situations involving $n \geq 3$ players and identified players' confidence degrees for equilibrium parametrization purposes. Marinacci (2000), on the other hand, gave more flexible definitions to players' vaguenesses on their beliefs, which could then be used in comparative statics studies. Klibanoff (1996) and Lo (1996) adopted Gilboa and Schmeidler's (1989) notion of ambiguity aversion and used convex and closed sets of probabilistic priors on products of other players' mixed strategies, reducible to those on their pure actions, as the basis on which players make decisions. Epstein (1997) let players be ambiguous about opponents' pure strategies as well as their ambiguity attitudes, and studied the iterated elimination of dominated strategies.

Players in the above games were allowed to have qualms about opponents' behaviors. We, like some studies of incomplete-information games involving general ambiguity attitudes, focus on the complementary situation where players have vagueness about factors external to all of them. We argue for merits of our focus as follows. First, as shown momentarily, mixed strategies chosen by players are often enforceable. Second, uncertainties about the state of the world can pose a much bigger problem than those about other players' behaviors. Think of a Stag Hunt game where each participant has only to choose between *cooperate* and *defect*, and yet there are millions of combinations in numbers, sizes, and speeds of the stags and hares on the hunting ground, as well as other factors like temperature and wind. Third, no longer having to model players' behaviors through non-probabilistic means, we can apply conventional tools built on countably additive probabilities to our analysis. Uncertainty about opponents' types will still