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Monotonicity and revenue equivalence domains by monotonic transformations in differences

Juan Carlos Carbajal[a], Rudolf Müller[b,*]

[a] School of Economics, UNSW Australia, Sydney, NSW 2052, Australia

[b] Business Intelligence and Smart Services Institute, Maastricht University, Maastricht, The Netherlands

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ABSTRACT

In a mechanism design setting with quasilinear preferences, a domain D of admissible valuations of an agent is called a monotonicity domain if every 2-cycle monotone allocation rule is truthfully implementable (in dominant strategies). D is called a revenue equivalence domain if every implementable allocation rule satisfies revenue equivalence. Carbajal and Müller (2015) introduced the notions of *monotonic transformations in differences* and showed that if D admits these transformations then it is a revenue equivalence and monotonicity domain. Here, we show that various economic domains, with *countable* or *uncountable* allocation sets, admit monotonic transformations in differences. Our applications include public and private supply of divisible public goods, multi-unit auction-like environments with increasing valuations, and allocation problems with externalities. Single-peaked domains admit only a modified version of monotonic transformations in differences. We show that this property implies too that single-peaked domains are revenue and monotonicity domains.

1. Introduction

In previous work (Carbajal and Müller, 2015), we considered allocation problems in a very general mechanism design setting with quasilinear utilities in which the agent's valuation function is private information. Treating the *preference domain* – the set of admissible valuation functions defined on an allocation set – as the primitives of the design problem, we provided a new set of sufficient attributes on it to ensure that any allocation rule that satisfies *2-cycle monotonicity* also satisfies *truthful (dominant strategy) implementability*. We also showed that these attributes guarantee that all implementable allocation rules satisfy the *revenue equivalence property*.

What distinguished our work from previous results available in the literature is that our conditions apply to finite *and* infinite allocation sets. To the best of our knowledge, the main results in Carbajal and Müller (2015) – see Theorems 1 and 2 in that paper – are the first in the literature to offer sufficient conditions for a revenue equivalence and monotonicity domain that apply to *finite non-convex* allocation sets with deterministic allocation rules and *infinite* allocation sets. The purpose of the current paper is to illustrate the breadth of these results with important economic applications not covered by previous work in the literature.

In Section 2, we introduce notation and some relevant definitions. Section 3 deals with uncountable allocation sets and is motivated by the literature on public good provision (Green and Laffont, 1977; Laffont and Maskin, 1980; Güth and Hellwig, 1986, among others), pollution rights (Dasgupta et al., 1980; Montero, 2008), and quasilinear exchange economies (Goswami et al., 2014). We prove that monotonic transformations in differences are admitted if A is an interval of the real line and the domain D consists of all continuous, non-negative, increasing functions. Monotonic transformations are also in place when the allocation set is a compact manifold and the domain of valuations is restricted to the set of all smooth functions. In all these cases, implementability and revenue equivalence are obtained from 2-cycle monotonicity. Unfortunately, the preference domain consisting of all concave valuations on a convex set does not satisfy monotonic transformations in differences around two alternatives.

We then extend the allocation set to be the product space of two subsets of the real line (at least one of which is finite). This allows us to model situations where externalities are present, as for instance models of technology licensing by an upstream monopolist to downstream competitors (Katz and Shapiro, 1986), or classic takeover models with atomistic stockholders (Grossman and Hart,

* Corresponding author.
E-mail addresses: jc.carbajal@unsw.edu.au (J.C. Carbajal), r.muller@maastrichtuniversity.nl (R. Müller).

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1980; Burkart et al., 1998), or models of diffusion of technology standards (Dybvig and Spatt, 1983). In all these cases, the valuation of the agent may be increasing in its first component (e.g., access to innovation) but decreasing in its second component (e.g., the number of competitors licensing the innovation). We show that here as well the domain of valuations allows for monotonic transformations in differences, thus every 2-cycle monotone allocation rule is implementable and satisfies revenue equivalence.

The setting we study in Section 4 resembles those considered in multi-unit auction problems (e.g., Dobzinski and Nisan, 2015) or in other allocation problems with indivisible goods. Monotonic transformations are present when the allocation set is a countable ordered set and the domain of valuations consists of all increasing valuations on A. Single-peaked domains do not admit monotonic transformations around two alternatives. However, Mishra et al. (2014) have shown that single-peaked domains are revenue equivalence and monotonicity domains. Inspired by their work, we introduce the notion of pairwise monotonic transformations in differences, which require distortions around one alternative alone for pairs of consecutive allocations in A. We then proceed to combine pairwise monotonic transformations with one of our prior conditions to obtain an analogue of Theorem 1 for this discrete, ordered allocation set. Using this new result, we are able to show that in single-peaked domains and truncated domains every 2-cycle monotone allocation rule is implementable and satisfies revenue equivalence.

2. Notation and definitions

Consider a general mechanism design setting with an arbitrary, nonempty allocation set A. There is a single agent[1] with quasi-linear preferences over alternatives and monetary transfers: his utility from choosing $a \in A$ and paying $\rho \in \mathbb{R}$ is $v(a) - \rho$. Instead of introducing a type space into the model, we treat the *valuation function* $v: A \to \mathbb{R}$ as the agent's private information. We refer to the set of admissible valuations $D \subseteq \mathbb{R}^A$ as the *preference domain*. To ease the notational burden, let $\triangle v(a,b) \equiv v(a) - v(b)$ represent the value difference between a and b under v, for all $a, b \in A$, $v \in D$.

In this setting, an *allocation rule* is a function $f: D \to A$. Let $f^{-1}(a) \subseteq D$ be the set of valuations that choose a under f — assume that $f^{-1}(a) \neq \emptyset$, for all $a \in A$. The allocation rule f is said to be *truthfully implementable* if there is a payment rule $\pi: D \to A$ such that

$$\triangle v(f(v), f(w)) \geq \pi(v) - \pi(w), \quad \text{for all } v, w \in D.$$

Defining the f-length between two alternatives $a, b \in A$ by

$$\ell_f(a,b) \equiv \inf\{\triangle v(a,b) : v \in f^{-1}(a)\},$$

f is said to be *2-cycle monotone* if for every 2-cycle $\{x, y, x\}$ in A,

$$\ell_f(x,y) + \ell_f(y,x) \geq 0. \tag{1}$$

The allocation rule f is said to be *cyclically monotone* if and only if for every integer $k \geq 2$, every k-cycle $\{a_1, a_2, \ldots, a_k, a_{k+1} = a_1\}$ in A has non-negative f-length; i.e.,

$$\sum_{i=1}^{k} \ell_f(a_i, a_{i+1}) \geq 0. \tag{2}$$

Clearly, f is truthfully implementable only if f is 2-cycle monotone, but 2-cycle monotonicity does not necessarily imply implementability in every allocation problem (D, f).[2] The relevance of the stronger cyclic monotonicity condition is that, as Rochet (1987) showed, it always characterizes truthful implementability. An important strand of the mechanism design theory has been devoted to finding conditions on the domain D under which 2-cycle monotonicity is equivalent to cyclic monotonicity.[3]

A second strand of the mechanism design theory has devoted efforts to understanding when the allocation rule fixes the incentive compatible payment rule, if any. Note that if f is implementable, using the Taxation Principle we can associate the incentive compatible payment rule π with a non-linear *price scheme* $p: A \to \mathbb{R}$ defined by

$$p(a) \equiv \pi(v), \quad \text{for all } v \in f^{-1}(a), \text{ for all } a \in A.$$

Say that f satisfies *revenue equivalence* if for all price schemes $p, q: A \to \mathbb{R}$ that implement it one has

$$p(a) - p(b) = q(a) - q(b), \quad \text{for all } a, b \in A.$$

It has been known for a while that when one can represent an incentive compatible price scheme solely by means of the allocation rule — for instance, as in Myerson's (1981) optimal auction design paper — revenue equivalence is in place.[4] Such representations require however valuations parameterized by types, and valuations for outcomes that behave analytically well as functions of types, for example, by being convex. Heydenreich et al. (2009) offered a characterization of revenue equivalence that relies solely on f-length, and use this to identify domains for which all truthfully implementable rules satisfy revenue equivalence.

A domain D is called a *monotonicity domain* if every 2-cycle monotone allocation rule is truthfully implementable. It is called a *revenue equivalence domain* if every truthfully implementable allocation rule satisfies the revenue equivalence property. Taken as given the desirability of understanding economic domains that admit well-behaved mechanisms, in Carbajal and Müller (2015) we provided two sufficient conditions on the preference domain to be a monotonicity and revenue equivalence domain.[5]

Definition 1. A domain D admits *bounded monotonic transformations in differences around one alternative (MD1)* if for all $x, y \in A$, $x \neq y$, for all $w \in D$ and all $\epsilon > 0$, there is a valuation $v \in D$ such that for every alternative $a \in A \setminus \{x\}$,

$$\triangle v(x,a) > \triangle w(x,a),$$

and the transformation v can be chosen to satisfy

$$\triangle v(x,y) < \triangle w(x,y) + \epsilon.$$

Definition 2. A domain D admits *monotonic transformations in differences around two alternatives (MD2*)* if for all $x, y \in A$, $x \neq y$, all $v^x, v^y \in D$, and essentially all $\delta \in \mathbb{R}$ satisfying $\triangle v^x(x,y) > \delta > \triangle v^y(x,y)$, there is a valuation $v \in D$ such that $\triangle v(x,y) = \delta$ and for each alternative $a \in A \setminus \{x, y\}$, either

$$\triangle v(x,a) > \triangle v^x(x,a) \quad \text{or} \quad \triangle v(y,a) > \triangle v^y(y,a).$$

D admits *bounded monotonic transformations in differences around two alternatives (MD2)* if in addition for all distinct $x, y, z \in A$, all $v^x, v^y \in D$, all $\epsilon > 0$ and essentially all $\delta \in \mathbb{R}$ such that

[1] Our results here extend to multi-agent environments without much problem when the solution concept is dominant strategy implementation.

[2] See Bikhchandani et al. (2006a) for a counter example.

[3] Among others, see Bikhchandani et al. (2006b), Saks and Yu (2005); Archer and Kleinberg (2014), Ashlagi et al. (2010) and Mishra et al. (2014). Note all these papers consider finite allocation sets.

[4] See also Williams (1999), Milgrom and Segal (2002), Chung and Olszewski (2007), Kos and Messner (2013) and Carbajal and Ely (2013).

[5] See Carbajal and Müller (2015) for an extensive discussion of the MD1 and MD2 conditions and for comparison of Theorem 1 with previous results.