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Chaotic dynamics in a three-dimensional map with separate third iterate: The case of Cournot duopoly with delayed expectations

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A B S T R A C T

We consider a Cournot duopoly with isoelastic demand function and constant marginal costs. We assume that both producers have naive expectations but one of them reacts with delay to the move of its competitors, due to a "less efficient" production process of a competitor with respect to its opponent. The model is described by a 3D map having the so-called "cube separate property", that is its third iterate has separate components. We show that many cycles may coexist and, through global analysis, we characterize their basins of attraction. We also study the chaotic dynamics generated by the model, showing that the attracting set is either a parallelepiped or the union of coexisting parallelepipeds. We also prove that such attracting sets coexist with chaotic surfaces, having the shape of generalized cylinders, and with different chaotic curves.

1. Introduction

In 2000, Bischi et al. [5] have studied a class of two-dimensional discrete maps having the property that their second iterate is a decoupled map. The essential result of such a study is that the dynamic properties of this kind of maps can be deduced from the simple analysis of a components of their second iterate, a one-dimensional map. More recently, Agliari et al. [4] have obtained analogous results considering a three-dimensional family of maps having third iterate with separate components. A typical feature of such maps is the coexistence of many invariant orbits, so that multistability situations are recurring. In the 3D case, for example, it is sufficient that the related 1D map has a cycle of period 2 to obtain two coexisting cycles for the starting map. And the same holds when chaotic dynamics are involved.

During the last years, many applications to Cournot duopoly have been studied, considering the quantity adjustment over time based on a two-dimensional map with separate second iterate (see, among others, [5,11,12]). Indeed "square separate maps" naturally arise when producers have naive expectations.

In the present paper, we consider a Cournot model in which producers have naive expectations about the production of the competitor, but one of them reacts with delay to the move of its competitor. From an economic point of view such an assumption can be justified by the fact that one of two competitors has a "slower" production process, meaning that its production process is technologically less advanced and consequently requires a longer time to react to the market demand. Stated in other words, we can say that a competitor has a production process based on a technology "less efficient" that its opponent.

Recently many authors have studied Cournot duopoly model with delay; in particular they have considered markets with memory, that is the expected quantity is a weighted average of the past quantity observations (see for example [6,7,13]). Here the framework is completely different, since we focus on the "delayed production" of a competitor.

The model under scrutiny is described by a discrete 2D *time-delayed system*, that can be rewrite as a 3D map M with the "cubic separate property" , that is, its third iterate has separated components. Our aim is to perform a global analysis of the model, characterizing the basin of attractions of the different coexisting attractors. Moreover, we shall extend the results in [4], deepening the study of the chaotic attractors of M. In particular, we shall analytically show that the three-dimensional chaotic sets (parallelepipeds) coexist with chaotic surfaces, given by union of generalized cylinders, and with chaotic curves.

The rest of the paper is organized as follows. In Section 2 we introduce the model describing the time evolution of the production levels of the two firms. We obtain a 3D map having third iterate with separate components. Then, for convenience of the reader, we recall some results achieved by Agliari et al. (see [4]) related to

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this kind of maps. In Section 3 we show that the model exhibits multiplicity of cycles and characterize their basins of attraction. In Section 4, extending the results of Agliari et al. [4], we study the chaotic attractors (parallelepipeds), showing that they coexist with many chaotic surfaces and curves. Section 5 concludes.

2. The model

We consider a Cournot duopoly in which two competitors produce perfect substitute goods. Following Puu [11], we assume an isoelastic demand function and constant marginal costs. Denoting by x and y the supplies of the competitors, the profits are given by

$$\Pi_1 = \frac{x}{x+y} - ax, \qquad \Pi_2 = \frac{y}{x+y} - by$$

where a and b are the constant marginal costs and, consequently, positive parameters.

The *Cournot–Nash reaction functions* of the two firms depend on the expected production of the opponent and are given by:

$$\begin{aligned} r_1\left(y^{(e)}\right) &= \arg\max_x \Pi_1\left(x, y^{(e)}\right) = \sqrt{\frac{y^{(e)}}{a}} - y^{(e)} \\ r_2\left(x^{(e)}\right) &= \arg\max_y \Pi_2\left(x^{(e)}, y\right) = \sqrt{\frac{x^{(e)}}{b}} - x^{(e)} \end{aligned} \tag{1}$$

defined in $\mathbb{R}_+$.

We conjecture that the firms adopt a “learning by doing” approach. Then in the time evolution of the production decisions, we assume that at each stage firms optimally decide following their reaction function, supposing that the production of the opponent remains the same. This means that the firms have *naive expectations*. Furthermore, we assume that a producer reacts with *delay* to the move of its opponent:

$q^{(e)}_{1,t} = q_{2,t-1}$ and $q^{(e)}_{2,t} = q_{1,t-2}$

In this way we obtain a discrete two dimensional *time-delayed system:*

$$\begin{cases} x_{t+1} = \sqrt{\frac{y_t}{a}} - y_t \\ y_{t+1} = \sqrt{\frac{x_{t-1}}{b}} - x_{t-1} \end{cases} \tag{2}$$

By means of the auxiliary variable $z_t = x_{t-1}$, we can rewrite (2) as a three dimensional model:

$$T : \begin{cases} x' = \sqrt{\frac{y}{a}} - y \\ y' = \sqrt{\frac{z}{b}} - z \\ z' = x \end{cases} \tag{3}$$

where the symbol ’ denotes the unit time advancement operator, that is, if (x, y, z) represents the vector of choices at time t, then (x', y', z') gives the choices at time $t+1$.

It is easy to see that the map (3) is not defined in the whole three dimensional phase-space. In fact the domain of T is the region D given by:

$$D = \{(x, y, z) : \ \ y \geq 0, z \geq 0\}$$

We are interested in a subset of D, denoted by S, which consists in the points (x, y, z) for which we have $T^n(x, y, z) \in D$, for any $n \geq 0$. We shall call *admissible* such points and trajectories in S:

$$S = \{(x, y, z) \in D : T^n(x, y, z) \in D \ \forall n \geq 0\}.$$

Before starting the analysis of the map (3) in the phase-space D we show that the two marginal costs (a, b) are redundant parameters. This follows from the observation that the maps T with parameters (a, b) and $\widetilde{T}$ with parameters $(\tau a, \tau b)$ with $\tau > 0$ are *topologically conjugate* via the homeomorphism $\Phi(x, y, z) = (\tau x, \tau y, \tau z)$, being $T = \Phi \circ \widetilde{T} \circ \Phi^{-1}$ or, equivalently, $\widetilde{T} = \Phi^{-1} \circ T \circ \Phi$. In the present model we have:

$$\widetilde{T} : \begin{cases} x' = \sqrt{\frac{y}{\tau a}} - y \\ y' = \sqrt{\frac{z}{\tau b}} - z \\ z' = x \end{cases}$$

Considering $\tau = \frac{1}{a}$ and setting $k = \frac{b}{a}$, we obtain the map:

$$M : \begin{cases} x' = \sqrt{y} - y \\ y' = \sqrt{\frac{z}{k}} - z \\ z' = x \end{cases} \tag{4}$$

Due to topological conjugacy, the dynamics of the map T which depends on two parameters (a, b) and the dynamics of the map M which depends on a unique parameter, k, have the same qualitative behavior, because they are associated by a simple coordinate transformation (see, [2]). In other words, through the topological conjugacy we have obtained a *reduction* in the number of parameters of the map T in (3), since only the ratio between marginal costs has to be considered.

Henceforth, we take the analysis considering the map M in (4) and $k \in (0, 1]$, assuming that producer 1 has higher marginal costs than its competitor. From an economic point of view, this choice can be justified assuming that the slow technology of producer 2 causes delayed reaction to the moves of its opponent but it is less expensive.

2.1. Properties of “cube separate maps”

Related to the aim of the present paper, the fundamental property of the map M in (4) is that *its third forward iterate has separate components,* which we will call “*cube separate property*”. Indeed the map in (4) belongs to the particular class of maps:

$$\Psi : \begin{cases} x' = f(y) \\ y' = g(z) \\ z' = h(x) \end{cases} \tag{5}$$

and $\Psi^{3n}(x, y, z) = (H^n(x), F^n(y), G^n(z))$, for each integer $n \geq 0$, with $H(x) = f(g(h(x)))$, $F(y) = g(h(f(y)))$, $G(z) = h(f(g(z)))$ and F^0, G^0, H^0 identity functions.

The family of map (5) has been studied by Agliari et al. (see, [3,4]) and, for convenience of the reader, we recall here some results, useful for the understanding of the subsequent analysis of the model. These results are based on the following relationships: for any $n \geq 1$ the three one dimensional (1D) maps H, F and G satisfy:

- $h \circ H^n(x) = G^n \circ h(x)$
- $g \circ G^n(z) = F^n \circ g(z)$
- $f \circ F^n(y) = H^n \circ f(y)$ and
- $g \circ h \circ H^n(x) = F^n \circ g \circ h(x)$
- $f \circ g \circ G^n(z) = H^n \circ f \circ g(z)$
- $h \circ f \circ F^n(y) = G^n \circ h \circ f(y)$.

Such properties imply that the invariant sets of the 1D maps H, G and F are strictly correlated. As an example, we have that any n–cycle of the map H, $\{x_1, x_2, \ldots, x_n\}$ admits *conjugated cycles*, given by a n–cycle of the map G, i.e. $\{z_1, z_2, \ldots, z_n\} = \{h(x_1), h(x_2), \ldots, h(x_n)\}$, and a n–cycle of the map F, i.e. $\{y_1, y_2, \ldots, y_n\} = \{g(h(x_1)), \ldots, g(h(x_n))\}$.[1] We remark that conjugated cycles have all the same stability property, since their multipliers are equal.

In the following we shall consider the map H to study the attractors of Ψ as well as their stability properties.

[1] Obviously, the same result holds starting from a cycle of G (or F).