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The increase in the resolvent energy of a graph due to the addition of a new edge

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ABSTRACT

The resolvent energy $ER(G)$ of a graph G on n vertices whose adjacency matrix has eigenvalues $\lambda_1, \ldots, \lambda_n$ is the sum of the reciprocals of the numbers $n-\lambda_1, \ldots, n-\lambda_n$. We introduce the resolvent energy matrix $\mathbf{R}(G)$ and present an algorithm that produces this matrix. This algorithm may also be used to update $\mathbf{R}(G)$ when new edges are introduced to G. Using the resolvent energy matrix $\mathbf{R}(G)$, we determine the increase in the resolvent energy $ER(G)$ of G caused by such edge additions made to G. Moreover, we express this increase in terms of the characteristic polynomial of G and the characteristic polynomials of three vertex-deleted subgraphs of G.

1. Introduction

Let G be a simple graph on n vertices having vertex set $\mathcal{V}(G) = \{1, 2, 3, \ldots, n\}$ and edge set $\mathcal{E}(G)$. Two vertices u and v are adjacent in G if and only if $\{u, v\} \in \mathcal{E}(G)$. If $\{u, v\} \notin \mathcal{E}(G)$, then the graph $G + uv$ is the graph with vertex set $\mathcal{V}(G)$ and edge set $\mathcal{E}(G) \cup \{\{u, v\}\}$. If H has the same number of vertices as G, then G is a proper subgraph of H if $\mathcal{E}(G) \subset \mathcal{E}(H)$. The graph $G - u$ denotes the one-vertex-deleted subgraph of G obtained from G after removing vertex u and the edges incident to u. The graph $G - u - v$ denotes the two-vertex-deleted subgraph $(G - u) - v$ of G.

Let $\mathbf{A}$ be the $n \times n$ adjacency matrix of G. The graph G has characteristic polynomial $\phi(G, x) = \det(x\mathbf{I} - \mathbf{A})$, where $\mathbf{I}$ is the identity matrix. The roots of $\phi(G, x)$ are the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\mathbf{A}$. The complete graph K_n on n vertices is the graph whose $n \times n$ adjacency matrix is $\mathbf{J} - \mathbf{I}$, where $\mathbf{J}$ is the matrix of all ones. On the other hand, the empty graph N_n on n vertices is the graph whose adjacency matrix is the $n \times n$ zero matrix.

A walk of length ℓ in G is a sequence of vertices $v_0, v_1, \ldots, v_\ell$ of G such that $\{v_i, v_{i+1}\} \in \mathcal{E}(G)$ for all $i \in \{0, 1, 2, \ldots, \ell - 1\}$. Such a walk is closed if $v_0 = v_\ell$. The kth spectral moment $M_k(G)$ of G is the sum of the kth powers of all of the eigenvalues of its adjacency matrix. Since $\mathrm{tr}(\mathbf{M})$, the trace of a matrix $\mathbf{M}$, is equal to the sum of the eigenvalues of $\mathbf{M}$ [21], $M_k(G) = \mathrm{tr}(\mathbf{A}^k)$. Moreover, it is well known that the entry in the jth row and kth column of $\mathbf{A}^\ell$ is equal to the number of walks of length ℓ in G, starting from $j \in \mathcal{V}(G)$ and ending at $k \in \mathcal{V}(G)$ [7]. Thus, $M_k(G)$ may be thought of as being the total number of closed walks of length k in G, starting and ending at any vertex.

In 1978, Ivan Gutman, motivated by research on the total π-electron energy of molecules, defined the *graph energy* [15] as $\sum_{i=1}^{n} |\lambda_i|$. Starting from 2006, a surprisingly high number of graph energy variants were proposed in the literature, each with their own applications. This 'energy deluge' is discussed in reference [16], which additionally surveys and compares several of these graph energy variants. For a more thorough discussion of many such alternative graph energies, the reader

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is referred to the books [20,22]. Moreover, in a recent paper [24], new upper bounds were produced for several of these graph energies.

One of the more recent of these graph energy variants, the *resolvent energy*, was introduced in [19], following the earlier works by Estrada and Higham [12], and Chen and Qian [5]. It is defined by

$$ER(G) = \sum_{i=1}^{n} \frac{1}{n-\lambda_i}.$$

Eventually, the resolvent energy was extensively studied [1,9,14,17,18]. Also, its Laplacian spectrum version was recently put forward [3,25].

In [19, Theorem 2], it was shown that

$$ER(G) = \sum_{k=0}^{\infty} \frac{M_k(G)}{n^{k+1}}.$$

Thus, the resolvent energy belongs to a general class of cumulative vertex centrality measures based on closed walks, originally put forward by Estrada and Higham in [12]. This class contains graph invariants of the form

$$\mathbb{E}(G) = \sum_{k=0}^{\infty} c_k M_k(G) \tag{1}$$

with the sequence of positive real numbers $c_0, c_1, c_2, \ldots$ chosen such that the Maclaurin series $\sum_{k=0}^{\infty} c_k x^k$ converges to some function $f(x)$. Since $M_k(G) = \mathrm{tr}(\mathbf{A}^k)$, we have the relation

$$\mathbb{E}(G) = \mathrm{tr}\big(f(\mathbf{A})\big).$$

For instance, when $-n < x < n$, the series $\sum_{k=0}^{\infty} n^{-k-1}x^k$ converges to $(n-x)^{-1}$. Since the eigenvalues of $\mathbf{A}$ also satisfy this inequality for any graph G (see, for example, [26]), the summation $\sum_{k=0}^{\infty} n^{-k-1}\mathbf{A}^k$ converges to $(n\mathbf{I}-\mathbf{A})^{-1}$. Note that the eigenvalues of $(n\mathbf{I}-\mathbf{A})^{-1}$ are $\frac{1}{n-\lambda_1}, \ldots, \frac{1}{n-\lambda_n}$, all of which are positive real numbers; hence, this inverse matrix exists for all graphs and is positive-definite. The resolvent energy $ER(G)$ is thus $\mathbb{E}(G)$ with $c_k = \dfrac{1}{n^{k+1}}$ for all k and with $f(x) = \dfrac{1}{n-x}$. The following lemma is consequently inferred.

Lemma 1.1. $ER(G) = \mathrm{tr}\big((n\mathbf{I}-\mathbf{A})^{-1}\big)$.

Two other particular cases of graph invariants pertaining to the class $\mathbb{E}(G)$ of the form (1) are the *Estrada index* [4,8,10,11,13], in which

$$c_k = \frac{1}{k!} \text{ for all } k,\ f(x) = e^x,\ \mathbb{E}(G) = EE(G) = \mathrm{tr}\big(e^{\mathbf{A}}\big)$$

and the *resolvent Estrada index* [2,5,12] (defined for graphs that are not complete) in which

$$c_k = \frac{1}{(n-1)^k} \text{ for all k, } f(x) = \frac{n-1}{n-1-x},$$
$$\mathbb{E}(G) = EE_r(G) = (n-1)\,\mathrm{tr}\big((\,(n-1)\mathbf{I}-\mathbf{A})^{-1}\big).$$

Clearly, there is a relation between the resolvent Estrada index $EE_r(G)$ and the resolvent energy $ER(G)$. Indeed, they are both based on the *resolvent matrix* of $\mathbf{A}$, defined by $(z\mathbf{I}-\mathbf{A})^{-1}$, where z is a complex variable [28]. The resolvent matrix of $\mathbf{A}$ exists for values of z that are not eigenvalues of $\mathbf{A}$.

It is clear, by Lemma 1.1, that studying the matrix $(n\mathbf{I}-\mathbf{A})^{-1}$ should elucidate research on the resolvent energy. Because of this, we first establish strict bounds for the entries of the matrix $(n\mathbf{I}-\mathbf{A})^{-1}$ in Section 2. Subsequently, we consider how this matrix changes after introducing a new edge to a graph G, leading to the algorithm in Section 4 that evaluates the resolvent energy of any graph without the need of evaluating any matrix inverse or any eigenvalues. In Section 5, the resolvent energy change δ caused by the introduction of a new edge in G is quantified using entries of $(n\mathbf{I}-\mathbf{A})^{-1}$. After deriving expressions for the entries of this matrix in terms of four characteristic polynomials related to G, we present a formula in Section 7 that evaluates δ from these characteristic polynomials.

2. The resolvent energy matrix

Motivated by the previous introductory section, we start this section by making the following definition.

Definition 2.1. The *resolvent energy matrix* of a graph G on n vertices having adjacency matrix $\mathbf{A}$ is the matrix $\mathbf{R}(G) = (n\mathbf{I}-\mathbf{A})^{-1}$.

We denote the resolvent energy matrix $\mathbf{R}(G)$ of Definition 2.1 by $\mathbf{R}$ if the graph G is clear from the context. Note that $\mathbf{R}$ has rational entries, since it is the inverse of a matrix with integer entries. Because of this, $ER(G) = \mathrm{tr}(\mathbf{R}(G))$ is a rational