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A closed-form solution to the Ramsey model with logistic population growth

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ABSTRACT

In this paper, we consider the Ramsey growth model with CIES utility function, Cobb–Douglas technology, and logistic-type population growth law. We show the model to have a unique non-trivial steady-state equilibrium (a saddle point) and prove the optimal path to be non-monotonic over time. Moreover, we derive a closed-form solution for the case where capital's share is equal to the reciprocal of the intertemporal elasticity of substitution.

1. Introduction

The Ramsey growth model is a neoclassical model of economic growth based primarily on the work of the economist and mathematician Frank P. Ramsey (1928), who was the first in the long history of economics to introduce calculus of variation to examine the question of how much a country would need to save and invest in order to maximize welfare. His ideas were later taken up independently by Cass (1965) and Koopmans (1965), and have now become a major workhorse model in modern macroeconomics. Recently, Accinelli and Brida (2007) have explored the implications of studying the Ramsey model within a framework where the change over time of the labor force is governed by the logistic growth law (for a more general notion of population than the logistic law see Guerrini, forthcoming). The resulting model happens to be described by a three dimensional dynamical system, whose unique non-trivial steady-state equilibrium is saddle-point stable. According to the Grobman–Hartman theorem (see Guckenheimer and Holmes, 1983), this local stability implies the preservation of the topological properties of the system under linearization in a neighborhood of the steady state. Since two eigenvalues are real and negative, there exists a plane of stability in which equilibrium paths converge to the steady state. We demonstrate that both consumption and capital do not decline monotonically toward their steady states. Many models of growth, including Ramsey model, have the property that the transitional dynamics are determined by a one dimensional stable manifold. As a consequence, all the variables converge to their respective steady states at the same constant speed, which is equal to the magnitude of the unique stable eigenvalue. By contrast, in the present model, the stable transitional path is a two dimensional locus, thereby introducing important flexibility to the convergence and transition characteristics. Finally, we derive a solution to this modified Ramsey model with constant elasticity preferences and Cobb–Douglas technology under the assumption that capital's share of GDP is equal to the reciprocal of the intertemporal elasticity of substitution. This assumption originates in a nice paper by Xie (1994), re-used in Boucekkine and Ruiz-Tamarit (2004), Wälde (2005), and later by Chilarescu (2008), Posch (2009), and Smith (2006, 2007). Note that Boucekkine and Ruiz-Tamarit (2008) relax this assumption, and Boucekkine et al. (2008) explore more deeply the properties of the closed-form solutions in terms of short-term dynamics. Moreover, Chilarescu's paper is incorrect in many aspects as clarified by Hiraguchi (2009a,b). A final comment. The parametric restriction that capital's share is equal to the reciprocal of the intertemporal elasticity of substitution implies a relatively high intertemporal elasticity of substitution above unity, which is likely to be a reasonable description of the real world. In fact, the traditional estimate of capital's share, which is about 0.03 (Simon, 1990), would imply the intertemporal elasticity of substitution to be as large as 3. Compared to usual average estimates of the intertemporal elasticity of substitution lying between 0 and 1 (e.g., Vissing-Jørgensen, 2002), this appears high. However, taking the capital's share in a broad sense, so that it includes human as well as physical capital, and increasing the capital's share, e.g., between 0.66 and 1 (Barro and Sala-i-Martin, 2004, suggest 0.75), then the intertemporal elasticity of substitution lies between 1 and 1.5. Attanasio and Vissing-Jørgensen (2003), Attanasio and Weber (1993), Bufman and Leiderman (1990), and Koskievic (1999) found values of the intertemporal elasticity of substitution in this range. Hence, with the capital's share between 0.66 and 1, the implied value for the intertemporal elasticity of substitution appears reasonable.

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2. The optimal control model

We consider the Ramsey growth model introduced by Accinelli and Brida (2007), where the economy system may be seen as a closed economy inhabited by many identical agents facing the following optimization problem

$$\max \int_0^\infty \frac{c_t^{1-1/\sigma}-1}{1-1/\sigma} e^{-\rho t} dt,$$

subject to

$$\dot{k}_t = k_t^\alpha - (\delta + a - bL_t)k_t - c_t. \tag{1}$$

In the instantaneous utility function, σ represents the constant intertemporal elasticity of substitution, $\rho > 0$ is the rate of time preference, and c_t is the real per capita consumption of a single good. Output y_t is produced using a stock of productive capital k_t according to the Cobb–Douglas production function $f(k_t) = k_t^a$, $\alpha \in (0,1)$. Capital depreciates at a constant rate $\delta > 0$. Population L_t evolves according to the law

$$\dot{L}_t = L_t(a - bL_t), a > b > 0. \tag{2}$$

This equation is called the Verhulst equation (Verhulst, 1838), and the underlying population model is known as the logistic model. As a function of time, it is a Bernoulli's differential equation. It is known that the change of variables $w = L^{-1}$ transforms it into a linear first-order differential equation in w, whose solution, assuming $L_0 = 1$, is easily found to be $L_t = \alpha e^{\alpha t}/(\alpha - b + be^{\alpha t})$. Notice that $L_\infty = \lim_{t\to\infty} L_t = a/b$. Solving this continuous-time dynamic problem involves using calculus of variations. The current-value Hamiltonian of this optimization problem writes as

$$H(k_t, c_t, L_t, \lambda_t) = \frac{c_t^{1-1/\sigma}-1}{1-1/\sigma} + \lambda_t [k_t^\alpha - (\delta + a - bL_t)k_t - c_t],$$

where λ_t is the costate variable associated to the budget constraint (1). The Pontryagin conditions for optimality are

$$H_{c_t} = 0 \Rightarrow c_t^{-1/\sigma} = \lambda_t, \tag{3}$$

$$\dot{\lambda}_t = \rho\lambda_t - H_{k_t} \Rightarrow \dot{\lambda}_t = -\lambda_t \left[\alpha k_t^{\alpha-1} - \rho - \delta - (a - bL_t)\right], \tag{4}$$

together with Eqs. (1) and (2), the boundary condition $k_0 > 0$, and the transversality condition $\lim_{t\to\infty} e^{-\rho t}\lambda_t k_t = 0$. Differentiating Eq. (3) with respect to time, and using formula (3), we can rid Eq. (4) of the $\dot{\lambda}_t$ and λ_t expressions. After rearrangement, we arrive at

$$\dot{k}_t = k_t^\alpha - (a - bL_t + \delta)k_t - c_t, \tag{5}$$

$$\dot{c}_t = \sigma c_t \left[\alpha k_t^{\alpha-1} - (a - bL_t) - \delta - \rho\right], \tag{6}$$

$$\dot{L}_t = L_t(a - bL_t). \tag{7}$$

These equations, together with the initial condition k_0, and the transversality condition

$$\lim_{t\to\infty} e^{-\rho t} c_t^{-1/\sigma} k_t = 0, \tag{8}$$

constitute the dynamic system which drives the economy over time.

3. Optimal dynamic path and local stability analysis

We now focus on the steady state, which is defined as a situation in which the growth rates of consumption, capital and population are zero. An asterisk below a variable will denote its stationary value.

Lemma 1. *The unique non-trivial steady state of the economy is*

$$k_* = \left(\frac{\alpha}{\delta + \rho}\right)^{\frac{1}{1-\alpha}}, \quad c_* = k_*^\alpha - \delta k_* = \frac{(1-\alpha)\delta + \rho}{\alpha} k_*, \quad L_* = \frac{a}{b}. \tag{9}$$

Proof. Eq. (9) is obtained from equating Eqs. (5)–(7) to zero, and solving the resulting system. □

We have the following result on the long-run behavior of the model's solution.

Lemma 2. *Let k_t and c_t be solutions of Eqs. (5) and (6), respectively. If there exists $\lim_{t\to\infty} k_t = k_\infty$, then k_∞ is finite and there exists $\lim_{t\to\infty} c_t = c_\infty < \infty$. Moreover, we have $(k_\infty, c_\infty) = (0,0)$, $(k_\infty, c_\infty) = (\delta^{1/(\alpha-1)}, 0)$ or $(k_\infty, c_\infty) = (k_*, c_*)$.*

Proof. Let $\tilde{k}_t$ be solution of the Solow differential equation $\dot{k}_t = k_t^\alpha - \delta k_t$. It is well-known that, as time passes, $\tilde{k}_t$ converges to its unique non-trivial steady state. Now, using the fact that $0 \le c_t$ and $0 \le \alpha - bL_t$, an application of a classical comparison theorem for ordinary differential equations (see Birkhoff and Rota, 1978) yields that $k_t \le \tilde{k}_t$. Consequently, $k_\infty < \infty$. By Lemma 1 of Guerrini (forthcoming), this fact allows us to derive that $\dot{k}_\infty = 0$. Hence, taking t to infinity in Eq. (5), and recalling that $L_\infty = a/b$, we find $c_\infty = \lim_{t\to\infty} c_t = k_\infty^\alpha - \delta k_\infty$, i.e. c_∞ exists finite. Applying again Lemma 1 of Guerrini (forthcoming), we get $\dot{c}_\infty = 0$ i.e. $(k_\infty^\alpha - \delta k_\infty)(\alpha k_\infty^{\alpha-1} - \delta - \rho) = 0$. If $k_\infty = 0$ or $k_\infty = \delta^{1/(\alpha-1)}$, then $c_\infty = 0$; if $k_\infty = [\alpha/(\delta + \rho)]^{1/(1-\alpha)} = k_*$, then $c_\infty = c_*$. □

Proposition 1. *The steady-state equilibrium (k_*, c_*, L_*) is a saddle point.*

Proof. Linearizing around the steady state yields the approximated dynamic system

$$\begin{bmatrix} \dot{k}_t \\ \dot{c}_t \\ \dot{L}_t \end{bmatrix} = J^* \begin{bmatrix} k_t - k_* \\ c_t - c_* \\ L_t - L_* \end{bmatrix}, \text{ with } J^* = \begin{bmatrix} \rho & -1 & bk_* \\ M & 0 & \sigma bc_* \\ 0 & 0 & -a \end{bmatrix},$$

having set $M = \sigma\alpha(\alpha-1)k_*^{\alpha-2}c_* < 0$. The Jacobian matrix J^* is obtained by differentiating the right-hand sides of Eqs. (5)–(7) with respect to the variables k_t, c_t, L_t, and evaluating them at the steady state. In order to characterize the local stability of the system, we need to compute the eigenvalues of J^*. It is immediate that one root of J^* is $\lambda_1 = -a < 0$, while the other two roots are the solutions of the equation $\lambda^2 - \rho\lambda + M = 0$, namely

$$\lambda_2 = \frac{\rho - \sqrt{\rho^2 - 4M}}{2} < 0, \quad \lambda_3 = \frac{\rho + \sqrt{\rho^2 - 4M}}{2} > 0.$$ □

In conclusion, we have found that J^* has one real positive (unstable) and two real negative (stable) roots. This proves that the steady state is (locally) a saddle point (Blume and Simon, 1994).

Proposition 1 is closely related to the classical Grobman–Hartman theorem (see Guckenheimer and Holmes, 1983), which states that around a hyperbolic equilibrium the qualitative properties of the non-linear systems (5)–(7) are preserved by its linearization. Our steady state is clearly hyperbolic because the Jacobian matrix calculated at that point has no zero or purely imaginary eigenvalues. According to Proposition 1, there exists a unique optimal path which asymptotically converges towards the steady state. Because of the transversality condition (8), the optimal path is restricted to the stable hyperplane,