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The pricing of liabilities in an incomplete market using dynamic mean–variance hedging

Robert J. Thomson *

School of Statistics and Actuarial Science, University of the Witwatersrand, Private Bag 3, WITS 2050, South Africa

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Abstract

In this article the method of pricing the liabilities of a financial institution by means of dynamic mean–variance hedging is applied to the situation of an incomplete market that is nevertheless in equilibrium with homogeneous expectations. For a given stochastic asset–liability model that is consistent with the market, the article shows how to determine the price at which, subject to specified provisos, a prospective transferor or transferee would be indifferent to the transfer of the liabilities.

MSC: C61; G20; IM12; IM22

Keywords: Market value of liabilities; Dynamic mean–variance hedging; Equilibrium market models; Incomplete markets

1. Introduction

Because of moral hazard, legal constraints and the de facto incompleteness of markets, it is generally impossible to replicate the liabilities of a financial institution with traded assets. Under such conditions it is impossible to determine the price at which the liabilities would be traded by means of asset matching, risk-neutral pricing methods or deflators (Møller, 2002; Jarvis et al., 2001). However, consider the case in which:

- the market, though incomplete, is in equilibrium;
- investors are non-satiated and risk-averse, have homogeneous expectations and make their choices in mean–variance space; and
- a stochastic asset–liability model (ALM) is adopted that is consistent with the market.

In this case, can a unique price be obtained that is consistent both with the ALM and with the market?

A number of authors (e.g. Martin and Tsui, 1999, p. 357) consider the price at which an asset or liability would trade if a complete market existed or (e.g. Cairns, 2001) the price at which an asset or liability would trade if a liquid market existed in it. In an incomplete market, however, extra risks exist, which cannot be hedged. Those risks may affect prices. The price contemplated

* Tel.: +27 11 646 5332; fax: +27 11 717 6285.
E-mail address: rthomson@icon.co.za.

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in this article is the price at which a prospective buyer or a seller who is willing but unpressured and fully informed would be indifferent about concluding the transaction, provided the effects of moral hazard and legal constraints would not be altered by the transaction. The price therefore allows for the situation in which the non-systematic risks of the liabilities cannot be hedged or diversified away. The case contemplated by Martin and Tsui (1999) is a special case of the general case considered in this article. A price based on the indifference of a prospective buyer or seller is not necessarily unique; the question whether the price obtained in this article is unique for the case defined above is revisited at the end of Section 2.

In practice, the determination of such a price is non-trivial. Hairs et al. (2002, pp. 273–274) suggest 'the selection of a replicating portfolio by minimising the asset–liability cash-flow mismatches over time'. Møller (2002, pp. 794–798) outlines solutions based on four different approaches in continuous time: the 'super-replication' approach, the 'utility' approach, the 'quadratic' approach and 'quantile hedging and shortfall-risk minimisation'. The 'quadratic' approach comprises two alternatives: 'risk minimisation' and 'mean–variance hedging'. Risk minimisation involves minimising a process reflecting the costs of financing a strategy that meets the cash flow exactly. Mean–variance hedging is largely attributable to Bouleau and Lamberton (1989), Duffie and Richardson (1991) and Schweizer (1992). It involves approximating the cash flow as closely as possible to the terminal value of a self-financing strategy so as to minimise the variance of the difference. These authors are interested in determining optimal hedging processes rather than in pricing. They all address the mean–variance hedging process in continuous time for claims contingent on share prices whose processes do not permit complete hedging due, for example, to jumps. The first assumes that the state space is a Markov process. The second and third assume that prices are geometric Brownian motions (the third more generally and rigorously than the second). Numerous subsequent papers have generalised their findings or applied them to particular cases.

Wise (1984a,b, 1987a,b, 1989), Wilkie (1985) and Keel and Müller (1995) explore the application of mean–variance portfolio theory to the liabilities of a financial institution in the one-period case. Schäl (1994), Schweizer (1995) and Černý (1999) derive the mean–variance hedging process in discrete time.

Mayers (1972) modifies the capital asset pricing model (CAPM) to allow for unmarketable assets, but the paper is concerned about the hedging and pricing of marketable assets in the presence of unmarketable assets, not about the pricing of the unmarketable assets themselves. By definition, the unmarketable assets are not part of the equilibrium market.

Mossin (1968) approaches the multiperiod optimisation of portfolios in discrete time using expected utility theory and dynamic programming. Stapleton and Subrahmanyam (1978) extend this analysis to include the establishment of equilibrium, conditional on information at the start of each period. Their analysis is also based on specified utility functions.

Svensson and Werner (1993) find that, in an incomplete market:

"the implicit value of [a] nontraded asset depends on both the nontraded assets themselves and the investor's preferences, and hence the value of a claim on future income is generally investor-specific."

They develop the differential equation for a non-traded asset in continuous time, based on a general preference function comprising a utility function of wealth discounted at a constant rate of liquidity preference. This result would be equally applicable to the pricing of liabilities in an incomplete market. They illustrate the application of the result to the case of a non-traded asset with constant drift and standard deviation and an exponential utility function, for which they obtain a solution in closed form.

Cairns (2001) also applies the utility approach to the pricing of liabilities. He assumes a single period with normally distributed returns and investors with exponential utility and heterogeneous expectations (i.e. different ALMs) and a market in equilibrium. He determines the price of the liability by considering the introduction into the market of an asset defined in the same way as the liability, and finding the price at which equilibrium is restored.

In this article, mean–variance hedging is applied to the liabilities of a financial institution rather than to a contingent claim (though the latter is included as a special case). The state-space vector is assumed to follow the form of a Markov process. As noted by Bouleau and Lamberton (1989), however, this includes some