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Harmonic mappings with hereditary starlikeness

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We study a hereditary starlikeness property for planar harmonic mappings on a disk and on an annulus. While such a property is a common trait of conformal mappings, it may be absent in harmonic mappings. It turns out that a sufficient condition for a harmonic mapping f to possess this hereditary property is to have a harmonic argument — a striking feature of conformal mappings that does not extend to all harmonic mappings.

1. Introduction

Harmonic mappings, which are complex-valued sense-preserving one-to-one functions satisfying Laplace's equation $\Delta f = 0$ on their respective domains in $\mathbb{C}$, possess some interesting properties. For instance, while it follows from a sharp result of Heinz [4, Lemma] that Euclidean and hyperbolic distances are not necessarily shortened by harmonic mappings of hyperbolic regions (see, e.g., [3, page 77] or [11, page 91]), the Lebesgue area measure of concentric disks

$$\mathbb{D}_r = \{z \in \mathbb{C}\colon\ |z| \le r < 1\}$$

is reduced by harmonic mappings preserving the unit disk $\mathbb{D}$ [11, Theorem 1.1].

If the image of the unit disk under a conformal mapping is a starlike region Ω, then the image of every disk in $\mathbb{D}$ is also starlike (see, e.g., [2, proof of Theorem 2.10]). This hereditary starlikeness property need not hold for harmonic mappings. For example, the harmonic mapping

$$f(z) = \operatorname{Re}\frac{z}{1-z} + i\operatorname{Im}\frac{z}{(1-z)^2}$$

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maps $\mathbb{D}$ onto the half plane $\{w\colon w > -\frac{1}{2}\}$, which is starlike, but $f(\mathbb{D}_r)$ is not starlike for $\sqrt{\frac{7\sqrt{7}-17}{2}} < r < 1$ [13, Example 1.1]. Nonetheless, we will provide sufficient conditions for harmonic mappings to possess this hereditary property. We will also establish analogous results for harmonic mappings between doubly-connected regions, which is the primary focus of this paper.

2. Preliminaries and main results

For $0 < \rho \le r < 1$, let $\mathbb{A}_\rho$ denote the annulus $\{z \in \mathbb{C}\colon \rho < |z| < 1\}$, let $\overline{\mathbb{A}_\rho}$ be its closure, and let $\mathbb{T}_r$ represent the circle $\{z \in \mathbb{C}\colon |z| = r\}$. We will use $\mathbb{T}$ to denote the unit circle $\partial\mathbb{D}$. A *curve* is a continuous image of the interval $[0, 1]$.

2.1. Starlikeness of simply and doubly connected regions

Let f be a harmonic mapping of D, where D is either $\mathbb{D}$ or $\mathbb{A}_\rho$. In the latter situation, we may assume without loss of generality that the inner and outer boundaries of D are mapped respectively to the inner and outer boundaries of $f(D)$, and it should be noted that the outer boundary of $f(D)$ necessarily contains points in $\mathbb{C}$ [7, Section 4.1]. Then the analytic characterization for $f(\mathbb{T}_r)$, where $0 < \rho < r < 1$, to enclose a starlike region S is the existence of a point $a \in S$ such that

$$\frac{\partial}{\partial t}\arg[f(re^{it}) - a] \ge 0. \tag{1}$$

Geometrically, $\arg[f(re^{it}) - a]$ increases as $\mathbb{T}_r$ is traced counterclockwise, and any ray emanating from a intersects $f(D)$ in a single, possibly infinite, line segment. It is standard terminology to say that S is starlike with respect to a. In particular, S is strictly starlike with respect to a if the inequality in (1) is strict. In geometrical terms, strict starlikeness with respect to a means that no tangent line to the boundary of S contains a. A normalization may achieved by applying a translation to S (or $f(D)$) so that $a = 0$, and we will define a curve to be *(strictly) starlike* if it forms the boundary of a region that is (strictly) starlike with respect to the origin. A *doubly-connected starlike region* is one whose intersection with any ray from the origin is either a single line segment or a single ray. If $D = \mathbb{D}$, then the representation $f = h + \bar{g}$ for some holomorphic functions g and h allows one to rewrite (1) when $a = 0$ as (see [1, page 139])

$$|h(z)|^2 \operatorname{Re}\frac{zh'(z)}{h(z)} \ge |g(z)|^2 \operatorname{Re}\frac{zg'(z)}{g(z)} + \operatorname{Re}[z(h(z)g'(z) - h'(z)g(z))].$$

Nevertheless, we will work directly with (1). Our first result is as follows.

Theorem 2.1. *Suppose f is a harmonic mapping of $\mathbb{D}$ onto a starlike region $\Omega_0 \subset \mathbb{C}$. Assume that on $\mathbb{A}_{\sqrt{2}-1}$,*

$$\Delta \operatorname{Im}\log f = 0, \tag{2}$$

where Δ represents the Laplace operator. Then $f(\mathbb{D}_r)$ is a strictly starlike region for $0 < r < 1$.

Remark 2.2. The boundary of a starlike region need not be a curve. For instance, the set $\left\{\left(2 + \sin\frac{1}{t}\right)e^{it}\colon t \in (0, 2\pi]\right\} \cup [1, 3]$ is the boundary of a starlike region, but is not locally connected, and is therefore not a curve by the Hahn–Mazurkiewicz theorem (see, e.g., [14, page 89]).

Given a harmonic mapping f of $\mathbb{D}$, where $f(\mathbb{D})$ is a starlike region with respect to the origin, it is known that $f(\mathbb{D}_r)$ is starlike with respect to the origin for at least $0 < r \le \sqrt{2} - 1$ (see, e.g., [13, Theorem 2.16(iii)]). This explains the restriction on (2) in Theorem 2.1 to the annulus $\mathbb{A}_{\sqrt{2}-1}$. It follows that Theorem 2.1 is a consequence of the more general result below.