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On the bi-harmonic maps with potential

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Abstract. In this note we characterize the harmonic maps and biharmonic maps with potential, and we prove that every biharmonic map with potential on a complete manifold satisfying some conditions is a harmonic map with potential.

Keywords: Harmonic maps with potential; Biharmonic maps with potential; Complete manifold; H-energy

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1. INTRODUCTION

The concept of harmonic maps with potential, was initially suggested by Ratto in [14] and recently developed by several authors : V. Branding [2], Y. Chu [5], A. Fardoun et al. [11], R. Jiang [12] and others.

In this paper we establish the second variation of the H-energy functional (Theorem 1), we introduce the notion of biharmonic maps with potential and we characterize the biharmonic maps with potential (Theorem 3). Also we prove that every biharmonic map with potential

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on a complete manifold satisfying some conditions is a harmonic map with potential (Theorem 5).

2. Harmonic maps with potential

Consider a smooth map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds, and let H be a smooth function on N. For any compact domain D of M the H-energy functional of φ is defined by

$$E_H(\varphi; D) = \int_D \big[e(\varphi) - H(\varphi)\big] v_g, \tag{2.1}$$

where $e(\varphi)$ is the energy density of φ defined by

$$e(\varphi) = \sum_i \frac{1}{2} h\big(d\varphi(e_i), d\varphi(e_i)\big), \tag{2.2}$$

v_g is the volume element and $\{e_i\}$ is an orthonormal frame on (M, g).

Definition 1 (*[14]*)**.** A map is called harmonic with potential H if it is a critical point of the H-energy functional over any compact subset D of M.

Let $\{\varphi_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth variation of φ supported in D. Then

$$\frac{d}{dt} E_H(\varphi_t; D)\Big|_{t=0} = -\int_D h(\tau_H(\varphi), v) v_g, \tag{2.3}$$

where $v = \frac{\partial \varphi_t}{\partial t}\Big|_{t=0}$ denotes the variation vector field of φ,

$$\tau_H(\varphi) = \tau(\varphi) + (\operatorname{grad}^N H) \circ \varphi, \tag{2.4}$$

and $\tau(\varphi)$ is the tension field of φ given by

$$\tau(\varphi) = \operatorname{trace} \nabla d\varphi = \sum_i \Big(\nabla^{\varphi}_{e_i} d\varphi(e_i) - d\varphi(\nabla^M_{e_i} e_i)\Big) \tag{2.5}$$

(see [1]).

Corollary 1 (*[10,14]*)**.** *A smooth map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds is harmonic with potential H if and only if*

$$\tau_H(\varphi) = 0.$$

Remark 1. Let $\varphi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds. If the potential H is constant, then φ is harmonic with potential H if and only if φ is harmonic map.
One can refer to [3,4,7–9] for background on harmonic maps and generalized harmonic maps.

2.1. *The second variation of the H-energy functional*

Theorem 1. *Let $\varphi : (M, g) \to (N, h)$ be a harmonic map with potential H between Riemannian manifolds and $\{\varphi_{t,s}\}_{t,s\in(-\epsilon,\epsilon)}$ be a two-parameter variation with compact support*