On the number of harmonic frames

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A B S T R A C T

There is a finite number $h_{n,d}$ of tight frames of n distinct vectors for $\mathbb{C}^d$ which are the orbit of a vector under a unitary action of the cyclic group $\mathbb{Z}_n$. These *cyclic harmonic frames* (or *geometrically uniform tight frames*) are used in signal analysis and quantum information theory, and provide many tight frames of particular interest. Here we investigate the conjecture that $h_{n,d}$ grows like n^{d-1}. By using a result of Laurent which describes the set of solutions of algebraic equations in roots of unity, we prove the asymptotic estimate

$$h_{n,d} \approx \frac{n^d}{\varphi(n)} \geq n^{d-1}, \qquad n \to \infty.$$

By using a group theoretic approach, we also give some exact formulas for $h_{n,d}$, and estimate the number of cyclic harmonic frames up to projective unitary equivalence.
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1. Introduction

Tight frames of n vectors for $\mathbb{C}^d$ have numerous applications (see the surveys [4], [5]). These include signal transmission with erasures [10], [14], [2] and quantum information theory [16], [20].

Many tight frames of practical and theoretical interest are G-frames (the orbit of a unitary action of a group G) [22]. Most notable are the *harmonic frames* (G is abelian) and *SICs*, i.e., d^2 equiangular lines in $\mathbb{C}^d$ (for a projective action of the abelian group $\mathbb{Z}_d^2$). The main result of this paper is a precise statement about how numerous the harmonic frames of n vectors for $\mathbb{C}^d$ are (Theorem 3.1). By way of comparison, SICs are known to exist only for certain values of d, and there is strong evidence for *Zauner's conjecture* that they exist for all values of d (see [20], [25]).

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We now provide some background on harmonic frames, and then detail our approach (precise definitions are given in §2). What we will call a *cyclic harmonic frame* for $\mathbb{C}^d$ was first introduced as a $d \times n$ submatrix $[v_1, \ldots, v_n]$ of the Fourier matrix (character table for $\mathbb{Z}_n$)

$$
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\
1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\
\vdots & \vdots & \vdots & & \vdots \\
1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)}
\end{bmatrix},
\qquad \omega := e^{\frac{2\pi i}{n}},
\tag{1.1}
$$

obtained by selecting d of the rows (characters of $\mathbb{Z}_n$). See [11], [13], [3] (who use the term *harmonic frame* for when the first d rows are taken), [6] (who use the term a *Fourier ensemble*), and [19] (who show a matrix with its columns given by a random cyclic harmonic frame is a RIP (restricted isometry property) matrix with high probability). These tight frames can be viewed as $\mathbb{Z}_n$-frames (called *geometrically uniform frames* in [9]). This construction generalises, with $\mathbb{Z}_n$ replaced by an abelian group G of order n [21], to give what we call a *harmonic frame* (it is *cyclic* if G can be taken to be $\mathbb{Z}_n$). It follows from the character table construction (and the fact there are a finite number of abelian groups of order n) that there is a finite number of harmonic frames of n vectors for $\mathbb{C}^d$.

A computer study [24] of the harmonic frames of n vectors for $\mathbb{C}^d$ suggested the following behaviour:

- The number of harmonic frames (up to unitary equivalence) grows like n^{d-1}, and it is influenced by the prime factors of n.
- The majority of harmonic frames are cyclic.

In this paper, we show that for fixed d the number $h_{n,d}$ of cyclic harmonic frames grows like

$$
h_{n,d} \approx \frac{n^d}{\varphi(n)} \geq n^{d-1}, \qquad n \to \infty.
$$

The key points of our argument are

- Cyclic harmonic frames correspond to d-element subsets $J \subset \mathbb{Z}_n$.
- When cyclic harmonic frames given by $J, K \subset \mathbb{Z}_n$ are unitarily equivalent, usually $K = \sigma J$ for some automorphism. When this is not the case, we say they are *exceptional*.
- The automorphisms of $\mathbb{Z}_n$ are easy to describe (as the units $\mathbb{Z}_n^*$).
- A pair of unitarily equivalent cyclic harmonic frames determines a torsion point on the $(2d)$-torus $\mathbb{T}^{2d}$.
- By using results about the torsion point solutions of algebraic equations, we show that the number of exceptional harmonic frames grows slower than the number which aren't.
- The nonexceptional cyclic harmonic frames are counted by Burnside enumeration.

We carry out this argument in §4–§5. We give examples and some numerical data in §6. In §7 we show that there are no exceptional equivalences when n is prime, and together with Burnside enumeration this allows us to give an exact formula for $h_{n,d}$ in this case, which we break down into lifted and unlifted, and into real and complex harmonic frames.

In the final section, we use our techniques to investigate the number $p_{n,d}$ of harmonic frames of n vectors for $\mathbb{C}^d$ up to *projective unitary equivalence*. For $d \geq 4$, this gives the lower estimate

$$
p_{n,d} \approx \frac{n^{d-1}}{\varphi(n)} \geq n^{d-2}, \qquad n \to \infty.
$$