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On closed manifolds with harmonic Weyl curvature

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We derive point-wise and integral rigidity/gap results for a closed manifold with harmonic Weyl curvature in any dimension. In particular, there is a generalization of Tachibana's theorem for non-negative curvature operator. The key ingredients are new Bochner–Weitzenböck–Lichnerowicz type formulas for the Weyl tensor, which are generalizations of identities in dimension four.

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1. Introduction

Let (M^n, g) be a closed manifold. R, W, Rc, E, S denote the Riemann curvature, Weyl tensor, Ricci tensor, traceable Ricci, and scalar curvature, respectively. In fact, the Riemann curvature decomposes into three orthogonal components, namely the Weyl curvature, a multiple of $\mathrm{E} \circ g$, and a multiple of $g \circ g$ ($\circ$ denotes the Kulkarni–Nomizu product). All could be seen as operators on the space of two-forms.

The study of the Einstein equation (Rc is constant) and its various curvature generalizations have a vast literature, see [4] for an overview. The Einstein condition says that the trace-less Ricci part vanishes and scalar curvature is constant; thus, in dimension at least four, the problem of understanding the Riemann curvature on Einstein manifolds reduces to investigate the Weyl tensor.

As a consequence, this paper aims to study a structure which generalizes the Einstein condition and involves the Weyl tensor. The outcomes are rigidity results.

For an Einstein manifold, S. Tachibana showed that nonnegative curvature operator implies locally symmetric [39]. Recently, S. Brendle improved the result by only assuming nonnegative isotropic curvature [7]. Those curvature assumptions are pointwise pinching W against scalar curvature. Similarly, integral pinching of W, in terms of its $L^{n/2}$-norm, was also investigated, such as [38,23,10].

A generalization of the Einstein condition is harmonic curvature: the former is $\mathrm{Rc} = \lambda g$ while the latter only requires Rc to be a Codazzi tensor. This condition is also of interests due to its connection to the theory of Yang–Mills equation. As a result, the topic has been studied in a wide array of literature, such as [15,25,21,41,16]. In particular, rigidity results due to integral pinched conditions were obtained in [21,16]. Also, A. Gray [18] deduced a classification under a point-wise assumption.

In connection with our interests, harmonic curvature is characterized by harmonic Weyl curvature (divergence of W is vanishing) and constant scalar curvature. Thus, it is natural to investigate the condition of harmonic Weyl curvature separately. In this area, most of the research has focused on the case of Kähler or dimension four. A Kähler manifold with that condition must have parallel Ricci tensor [18,40,30]. In dimension four, Weyl decomposes into self-dual and anti-self-dual parts ($\mathrm{W}^{\pm}$). A. Polombo [35] derived a Bochner–Weitzenböck formula for a Dirac operator and obtained applications for $\mathrm{W}^{\pm}$. Then, M. Micallef and M. Wang [31] proved that a closed four-dimensional manifold with harmonic self-dual and non-negative isotropic curvature (on the self-dual part) must be either conformally flat or quotient of a Kähler manifold with constant scalar curvature. As mentioned earlier, the curvature assumption is equivalent to the point-wise bound: for any eigenvalue ω of W^{+},