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Modified harmonic balance method for the solution of nonlinear jerk equations

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ABSTRACT

In this paper, a second approximate solution of nonlinear jerk equations (third order differential equation) can be obtained by using modified harmonic balance method. The method is simpler and easier to carry out the solution of nonlinear differential equations due to less number of nonlinear equations are required to solve than the classical harmonic balance method. The results obtained from this method are compared with those obtained from the other existing analytical methods that are available in the literature and the numerical method. The solution shows a good agreement with the numerical solution as well as the analytical methods of the available literature.

Introduction

Over the past decades, the study of nonlinear problems has been the interest of many researchers [1,2]. The exact solutions of these equations are rarely found. Therefore, researchers focused on numerical method [3] or analytical approximation methods [4] to solve those nonlinear problems. Das et al. [5] used a numerical method for the analysis of large amplitude beam vibration problem. Mohammed et al. [6] used mean monte-carlo finite difference method to study nonlinear epidemic system. Although these numerical methods are comparatively easy to program, they need heavy computational effort and proper initial guess values. In contrast, analytical approximation methods are more appealing because of their analytical expressions which are inherent in physical meanings and more suitable for parametric study.

There are many analytical approaches to construct the approximate periodic solutions of nonlinear differential equations. Perturbation methods are the most common and widely used technique for solving nonlinear differential equations, whereby the solution is expanded in the power series of a small parameter. Among the perturbation methods Lindstedt-Poincare (LP) method [7], Krylov-Bogoliubov-Mitropolisky (KBM) method [8–10] and multi-time expansion [11,12] are important. Though the perturbation methods provide an accurate result for small nonlinearity, it fails to provide desire results when nonlinearity becomes large.

To avoid the limitation of perturbation method several approximation methods such as iterative method [13,14], homotopy perturbation method [15–17], energy balance method [18] etc. are developed to determine the analytical approximate solution of the nonlinear differential equations. These methods are useful for both small and strong nonlinearity. Usually, a lower order (e.g., first or second) approximate solution is determined by these methods due to avoid algebraic complexities.

The harmonic balance (HB) method [19–27] is one of the most widely used technique for solving nonlinear differential equations, whereby the solution is expressed in the form of truncated Fourier series whose coefficients are determined by solving a set of algebraic equations. The method is not only valid for both weekly and strongly nonlinear differential equations, but also gives more accurate result than the other existing method. Several authors modified harmonic balance method to solve various types of nonlinear differential equations. Lau et al. [28] used incremental harmonic balance method to solve vibrational system. Wu et al. [29], introduce Newton harmonic balance method for solving a class of strongly non-linear oscillators. Alam et al. [30] introduced a new analytical technique based on classical harmonic balance method for solving nonlinear differential equations. Harmonic balance method is also used to solve various problems arise in electrical circuit such as microwave circuits [31] and computational fluid dynamics [32]. Recently, Hasan et al. [33] established the application of harmonic method by improving the computational capacity for forced vibration of nonlinear beams.

Although most of the dynamical systems are related to second-order differential equations, very few dynamical systems can be

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described by nonlinear third-order (nonlinear jerk) differential equations such as oscillations in a nonlinear vacuum tube circuit [34], thermo-mechanical oscillator in fluids [35]. In recent years, some researchers focused on the study of nonlinear jerk problems due to its various physical applications. The general form of the nonlinear jerk equation is $\dddot{x} = J(x, \dot{x}, \ddot{x})$. Following Gottlieb [36], the nonlinear jerk equation with only cubic nonlinearities has the form

$$\dddot{x} = J(x, \dot{x}, \ddot{x}) = -\gamma\dot{x} - \alpha\dot{x}^3 - \beta x^2\dot{x} + \delta x\dot{x}\ddot{x} - \varepsilon x\ddot{x}^2 \tag{1}$$

with the initial conditions

$$x(0) = 0, \dot{x}(0) = B, \ddot{x}(0) = 0 \tag{2}$$

where the over dot denotes the derivative with respect to time t, and the parameters γ, α, β, δ and ε are constants. Here, at least one of α, β, δ and ε should be non-zero. Gottlieb [36] used harmonic balance method to study the different cases of nonlinear jerk equation with cubic nonlinearity. Ma et al. [37] investigate the three cases of nonlinear jerk equation that are introduced by Gottlieb [36] using homotopy perturbation method. Leung et al. [38] presented residue harmonic balance method to solve nonlinear jerk equations.

In this paper, nonlinear jerk equations have been the subject of interest because of its increasing importance in physical problems modeling. The aim of the present article is to determine the approximate solutions of nonlinear jerk problems using modified harmonic balance method. In classical harmonic balance method, systems of nonlinear algebraic equations are solved to determine the unknown coefficient. However, in our present method the coefficients are expressed in terms small parameter for higher order approximation and only two nonlinear algebraic equations are required to solve for obtaining the results. The solution shows a good agreement with numerical results though the nonlinear term becomes significant.

The modified harmonic balance method

Let us consider a nonlinear differential equation

$$\dddot{x} + f(x, \dot{x}, \ddot{x}) = 0, \tag{3}$$

with the initial condition $x(0) = 0, \dot{x}(0) = B, \ddot{x}(0) = 0$

A periodic solution of Eq. (3) is chosen in the form

$$x = a\sin\varphi + a^3c_3\sin3\varphi + a^5c_5\sin5\varphi + \ldots \tag{4}$$

where $\varphi = \omega t$. a and $\dot{\varphi}$ are constants. In general, the unknown functions, $c_j, j = 3, 5, \ldots$ are determined together with a. Only sine terms are considered in solution of Eq. (3) because the coefficient of cosine terms is zero for the above initial conditions.

Now substituting Eq. (4) into Eq. (3) and expanding the function $f(x, \dot{x}, \ddot{x})$ in a Fourier series, we obtain

$$\begin{aligned} &-(a\cos\varphi + 27a^3c_3\cos3\varphi + 125a^5c_5\cos5\varphi + \ldots)\dot{\varphi}^3 \\ &+ [F_1(a, \dot{\varphi}, c_3, \ldots)\cos\varphi + F_3(a, \dot{\varphi}, c_3, \ldots)\cos3\varphi \\ &+ F_5(a, \dot{\varphi}, c_3, \ldots)\cos5\varphi + \ldots] = 0 \end{aligned} \tag{5}$$

By comparing the coefficients of equal harmonic, we obtain

$$a\dot{\varphi}^3 = F_1, \quad 27a^3c_3\dot{\varphi}^3 = F_3, \quad 125a^5c_5\dot{\varphi}^3 = F_5, \ldots \tag{6}$$

Utilizing the first equation of Eq. (6), we eliminate $\dot{\varphi}$ from all the rest. Thus Eq. (6) takes the following form

$$27a^2c_3F_1 = F_3, \quad 125a^4c_5F_1 = F_5, \ldots \tag{7}$$

We use a new parameter $\mu(\dot{\varphi}) \ll 1$ and express the equations of Eq. (7) in powers of μ as

$$c_j = c_{j,1}\mu + c_{j,2}\mu^2 + c_{j,3}\mu^3 + \ldots j = 3, 5, \ldots \tag{8}$$

Using the initial condition, we obtain

$$(a + 3a^3c_3 + 5a^5c_5 + \ldots)\dot{\varphi} = B \tag{9}$$

Finally, substituting the values of $c_3, c_5, \ldots$ into first equation of Eq. (6) and Eq. (9), we obtain a and $\dot{\varphi}$.

Examples

In this section, the modified harmonic balance method is applied to solve two cases of nonlinear jerk equation with cubic nonlinearity that is introduced by Gottlieb [36].

Jerk function of displacement time's velocity time's acceleration $x\dot{x}\ddot{x}$

Consider the Jerk equation in the following form

$$\dddot{x} + \dot{x} - x\dot{x}\ddot{x} = 0, \; x(0) = 0, \dot{x}(0) = B, \; \ddot{x}(0) = 0 \tag{10}$$

Let us consider a truncated form of Eq. (4) as

$$x = a\sin\varphi + a^3c_3\sin3\varphi + a^5c_5\sin5\varphi \tag{11}$$

Substitute Eq. (11) into Eq. (10), we have the following equation

$$\begin{aligned} &(a\dot{\varphi} - a\dot{\varphi}^3 + (a^3\dot{\varphi}^3 + 7a^5c_3\dot{\varphi}^3 + 18a^7c_3^2\dot{\varphi}^3 + 62a^9c_3c_5\dot{\varphi}^3 \\ &+57a^{11}c_3^2c_5\dot{\varphi}^3 + 50a^{11}c_5^2\dot{\varphi}^3)/4)\cos\phi \\ &+(3a^3c_3\dot{\varphi} - a^3\dot{\varphi}^3/4 - 27a^3c_3\dot{\varphi}^3 + 3a^5c_3\dot{\varphi}^3/2 + 27a^9c_3^3\dot{\varphi}^3/4 \\ &+21a^7c_5\dot{\varphi}^3/4 + 3a^9c_3c_5\dot{\varphi}^3/2 + 75a^{13}c_3c_5^2\dot{\varphi}^3/2)\cos3\phi \\ &+(5a^5c_5\dot{\varphi} - 13a^5c_3\dot{\varphi}^3/4 + 21a^7c_3^2\dot{\varphi}^3/4 - 125a^5c_5\dot{\varphi}^3 \\ &+5a^7c_5\dot{\varphi}^3/2 + 45a^{11}c_3^2c_5\dot{\varphi}^3/2 + 125a^{15}c_5^3\dot{\varphi}^3/4)\cos5\phi + \mathrm{HOH} = 0 \end{aligned} \tag{12}$$

where HOH stands for the higher order harmonics.

Comparing the coefficients of equal harmonics, we obtain

$$4 - \dot{\varphi}^2(4 - a^2 - 7a^4c_3 - 18a^6c_3^2 - 62a^8c_3c_5 - 57a^{10}c_3^2c_5 - 50a^{10}c_5^2) = 0 \tag{13a}$$

$$2c_3 - \dot{\varphi}^2(1 + 108c_3 - 6a^2c_3 - 27a^6c_3^3 - 21a^4c_5 - 6a^6c_3c_5 - 150a^{10}c_3c_5^2) = 0 \tag{13b}$$

$$20c_5 + \dot{\varphi}^2(-13c_3 + 21a^2c_3^2 - 500c_5 + 10a^2c_5 + 90a^6c_3^2c_5 + 125a^{10}c_5^3) = 0 \tag{13c}$$

With the help of Eq. (13a), eliminating $\dot{\varphi}^2$ from the Eq. (13b-c), we obtain Eq. (14a-b)

$$\begin{aligned} c_3 = \mu(&-4 + 12a^2c_3 - 84a^4c_3^2 - 108a^6c_3^3 + 84a^4c_5 \\ &+ 24a^6c_3c_5 - 744a^8c_3^2c_5 - 684a^{10}c_3^3c_5) \end{aligned} \tag{14a}$$

$$\begin{aligned} c_5 = \mu\Big(&-\frac{52}{5}c_3 + \frac{84}{5}a^2c_3^2 + 4a^2c_5 - 28a^4c_3c_5 - 248a^8c_3c_5^2 \\ &-228a^{10}c_3^2c_5^2 - 100a^{10}c_5^3\Big) \end{aligned} \tag{14b}$$

where $\mu = \frac{1}{384}$.

Using the initial condition, we obtain

$$(a + 3a^3c_3 + 5a^5c_5)\dot{\varphi} = B \tag{15}$$

Herein the four unknown quantities $\dot{\varphi}$, a, c_3, c_5 will be calculated from the four nonlinear equations (13a), (14a-b) and (15). According to the classical harmonic balance method, generally iteration or Newton-Raphson method is used to solve the nonlinear algebraic equations. In the present method the coefficients c_3, c_5 are expressed in the power series of a small parameter μ, and only two equations are solved by iteration method. Now let us consider

$$c_3 = c_{3,1}\mu + c_{3,2}\mu^2 + c_{3,3}\mu^3 + \ldots \tag{16a}$$