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Noncommutative harmonic analysis and image registration

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ABSTRACT

The image registration problem on a group G asks, given two functions $f, g\colon G \to \mathbb{R}$ that are related by a translation $f(x) = g(s^{-1}\cdot x)$ by an element $s \in G$, to find s. For abelian groups, the Fourier transform provides an elegant and fast solution to this problem. For nonabelian groups, the problem is much more involved. This paper shows how this applied problem can shed light on the constructions of noncommutative harmonic analysis, in particular the theory of Gelfand pairs. The abstract theory then suggests a novel two-step approach to solving such problems. The Gelfand pair $(\mathrm{SO}(3), \mathrm{SO}(2))$ then provides us with an intuitive solution of the registration problem for images on S^2.

1. Introduction

It does not happen very often that an abstract mathematical theory suddenly finds an application so intuitive that it can easily be explained to a lay person. This paper shows how the theory of Gelfand pairs from noncommutative harmonic analysis can help solve the image registration problem explained below. In a time when everybody is familiar with Photoshop, it might be interesting for a mathematician to think about a Gelfand pair in terms of image processing. And of course, as the marketing slogan goes, there is an App for that [7].

Images are functions on a two-dimensional domain, typically $\mathbb{R}^2$. Given two images f and g, the general image registration problem asks for a homeomorphic transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(x) = (g \circ T)(x) = g(T(x))$ for all $x \in \mathbb{R}^2$ or such that f is as close as possible to $g \circ T$. Imaging scientists have developed sophisticated algorithms with applications in medical and satellite imaging for very general transformations T, but in this paper, we restrict T to a more geometrically inspired problem. We want T to be an element of some low dimensional Lie group acting on $\mathbb{R}^2$.

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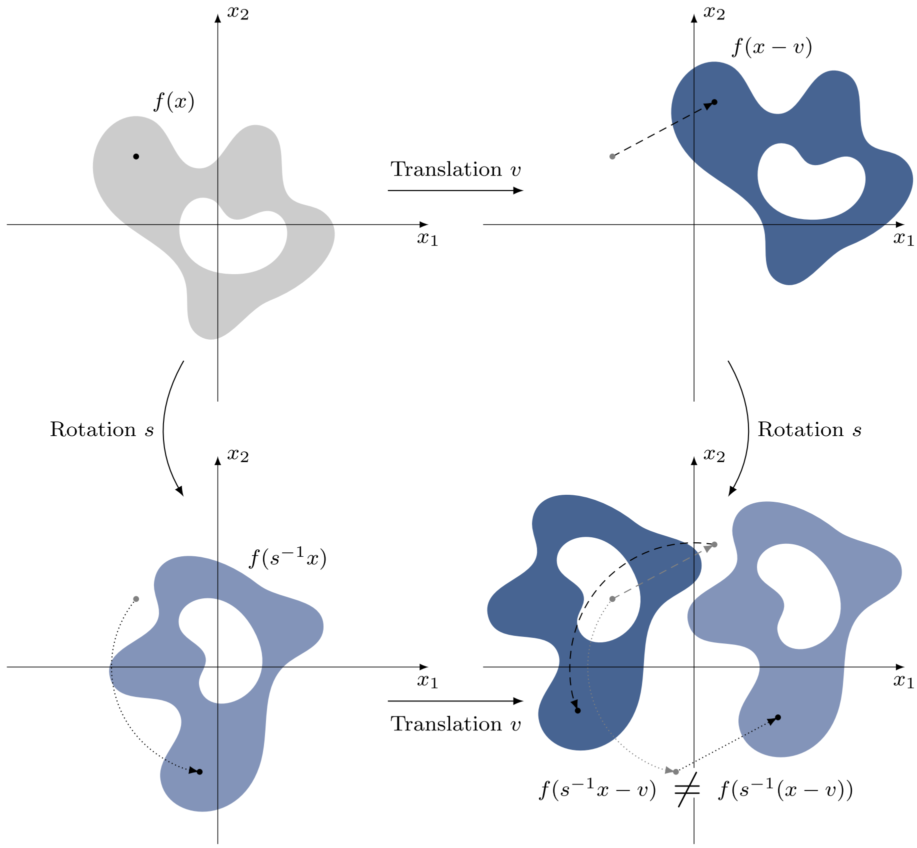

Fig. 1. The image registration problem with respect to the group of translations and rotations of the plane asks for the group element that brings two images into congruence. Horizontal arrows show translations, for which classical harmonic analysis provides a simple solution. Vertical arrows show rotations. The group of rotations and translations is not abelian, illustrated by the fact that the two paths through the diagram do not give the same picture. This noncommutativity is a major source of difficulty when trying to solve the registration problem.

The most basic classic image registration problem considers only translations of the plane. It asks for a vector v such that $f(x) = g(x+v)$. It has a very elegant solution using harmonic analysis which we will recall in section 2. Thanks to the Fast Fourier Transform (FFT), this solution is quite efficient and has many practical applications, although there are some technical drawbacks. It depends, however, on the availability of a rather rich mathematical structure. The functions form a Hilbert space $L^2_{\mathbb{C}}(\mathbb{R}^2)$ with the scalar product based on Lebesgue measure. Convolution provides a product of $L^2_{\mathbb{C}}(\mathbb{R}^2)$-functions with values in $L^1_{\mathbb{C}}(\mathbb{R}^2)$. The Fourier transform converts convolution products into ordinary pointwise products. All this is possible because the underlying group structure on $\mathbb{R}^2$ is abelian.

This paper explores image registration problems on other domains than just the plane and with more complicated transformation groups. Fig. 1 illustrates the problem for the group of rotations and translations of the plane. There are of course many algorithms in use for this type of problem. In astrophotography one can extract triangles or larger polygons formed by stars from the images, and compare them using suitable invariants. This is what http://astrometry.net has been doing for years, a very successful public service to locate uploaded images in the sky (see the presentation [10] for details).

We are interested in the question whether the abstract noncommutative harmonic analysis available for certain locally compact topological groups can help solve the image registration problem in a more general way. Our emphasis is not on providing the most direct route to a fully streamlined algorithm, but rather to show how the concept of the Gelfand pair developed in the abstract theory of noncommutative harmonic analysis suggests new solutions to this old and well known problem.

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