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Numerical artifacts in the Generalized Porous Medium Equation: Why harmonic averaging itself is not to blame

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ABSTRACT

The degenerate parabolic Generalized Porous Medium Equation (GPME) poses numerical challenges due to self-sharpening and its sharp corner solutions. For these problems, we show results for two subclasses of the GPME with differentiable $k(p)$ with respect to p, namely the Porous Medium Equation (PME) and the superslow diffusion equation. Spurious temporal oscillations, and nonphysical locking and lagging have been reported in the literature. These issues have been attributed to harmonic averaging of the coefficient $k(p)$ for small p, and arithmetic averaging has been suggested as an alternative. We show that harmonic averaging is not solely responsible and that an improved discretization can mitigate these issues. Here, we investigate the causes of these numerical artifacts using modified equation analysis. The modified equation framework can be used for any type of discretization. We show results for the second order finite volume method. The observed problems with harmonic averaging can be traced to two leading error terms in its modified equation. This is also illustrated numerically through a Modified Harmonic Method (MHM) that can locally modify the critical terms to remove the aforementioned numerical artifacts.

1. Introduction

We discuss how to discretize the self-sharpening Generalized Porous Medium Equation (GPME) without the temporal oscillations, the locking, and the lagging reported in previous works [1–4]. In these works, the numerical artifacts have been attributed to harmonic averaging, and arithmetic averaging has been proposed to resolve them. The exact causes of the artifacts were not published in the literature. In this paper, we aim to understand the exact causes. We demonstrate that harmonic averaging is not solely to blame and that the artifacts also depend on both the spatial and temporal discretizations. The critical terms are identified through modified equation analysis and numerical evidence is provided to show that counteracting these terms removes the artifacts. We refer to this demonstration method as the Modified Harmonic Method (MHM).

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The GPME is given in its multi-dimensional, conservative form as,

$$\begin{aligned} &p_t = \nabla \cdot (k(p)\nabla p) \ \text{ in } \ \Omega, \\ &p(x,0) = h(x), \\ &p(x,t) = g(x,t), \ \forall x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where $\partial\Omega$ is the boundary of the domain Ω, $h(x)$ is the initial condition and $g(x,t)$ is the Dirichlet boundary condition. The GPME can be seen as a subset of the fully general equation class that we are interested in, where $k(p)$ in Eqn. (1.1) is replaced by $k(p,x)$. The fully general equation class includes the variable coefficient problem, which will be discussed in a subsequent paper Maddix et al. [5].

1.1. Applications of the GPME

The Porous Medium Equation (PME), a subclass of the GPME, is used in a number of applications. In the PME,

$$k(p) = p^m, \tag{1.2}$$

for $m \geq 1$. The name PME originates from modeling gas flow in a porous medium. The PME can be derived from the continuity equation, Darcy's Law and the equation of state for perfect gases [6,7]. The polytropic exponent m relates the pressure $k(p)$ to the density p. The case $m = 1$ models isothermal processes, while $m > 1$ for adiabatic processes. From experimental data, Vázquez [6] found that $m = 1.405$ corresponds to airflow at normal temperatures.

Nonlinear heat transfer of plasmas (ionized gases), mainly by radiation, drove much of the initial theoretical research on the PME [6]. Zel'dovich and Raizer [8] proposed a model of this heat transfer at very high temperatures given by Eqn. (1.1). Here, the coefficient $k(p)$ is also a monomial of p where in this case, $k(p)$ represents the radiation thermal conductivity and p the temperature. In the optically thick limit approximation, $m = 3$. In more complex models with multiple ionized gases, this exponent can increase to 4.5–5.5.

The PME with $m = 1$ has several applications, including groundwater flow with Boussinesq's equation [9] and population dynamics in biology [6].

An example of the GPME that we will also consider is superslow diffusion [6,10] with

$$k(p) = \exp(-1/p). \tag{1.3}$$

The name superslow diffusion was introduced because the diffusivity $k(p) \to 0$ as $p \to 0$ faster than any power of p [11]. This equation is used to model the diffusion of solids at different absolute temperatures p. The coefficient $k(p)$ now represents the mass diffusivity and is connected with the Arrhenius law in thermodynamics [12].

In the above examples of the GPME, $k(p)$ is a continuous function with respect to p. An example of the GPME with discontinuous $k(p)$ is a Stefan problem [6], where $k(p)$ is a step function, e.g.

$$k(p) = \begin{cases} 1, & \text{if } p \geq 0.5, \\ 0, & \text{otherwise}. \end{cases} \tag{1.4}$$

The GPME with discontinuous coefficients is the topic of a subsequent paper in Maddix et al. [13].

In this paper, we focus on the GPME with continuous $k(p)$. In particular, we will use the PME and the superslow diffusion equation. For the PME, we consider the physical values for m up to 3, which is large enough to exhibit the numerical artifacts of interest.

1.2. Degeneracy of the GPME and its numerical challenges

Theoretical properties and behaviors of the GPME have been studied in many works, dating back to the 1950s with Barenblatt and Vishik [14], Barenblatt and Zel'dovich [15], Oleĭnik et al. [16], Kalašinkov [17] and Aronson [18] to more recently with Shmarev [19,20]. In [14,15], the Barenblatt–Prattle self-similar solution of the PME and its finite propagation property are derived. Self-similar solutions of the GPME have also been found [21]. Vázquez [6], Ngo and Huang [7] provide review papers and detailed references.

Some of the behaviors of the GPME may be surprising since at first glance it appears to be a straightforward variation of the heat equation. For example, self-sharpening can occur even with smooth initial data. This self-sharpening effect is illustrated in Fig. 1a for $k(p) = p^3$ ($m = 3$), in which a linear initial condition develops a sharp gradient over time. Fig. 1b depicts a moving interface that can be developed for compactly supported initial conditions. The speed of the interface can be calculated exactly with Darcy's Law [6,7]. In this solution, we see a sharp corner develop at the front. Because of these behaviors, the GPME is referred to as degenerate parabolic.

In Appendix A, we extend the PME results in [6] to the GPME and show that the governing equation (1.1) can be expressed in terms of $k(p)$ as