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Quasiconformal Harmonic mappings and the curvature of the boundary ☆

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We estimate the Jacobian of harmonic mapping of the unit disk onto a smooth and convex Jordan domain via the boundary function and the boundary curvature of the image domain. By using this result we make some asymptotically sharp estimates of constant of quasiconformality for harmonic diffeomorphisms between the unit disk and the convex domains via their boundary mappings.

Contents

1. Introduction and preliminary results

One of central questions on harmonic mapping theory is under what condition a homeomorphism F of the unit circle onto a Jordan curve γ generates, via Poisson integral a harmonic diffeomorphism. A fundamental result in this direction is the Rado–Choquet–Kneser theorem which asserts that, if γ is convex and F is a homeomorphism, then $w = P[F]$ is a diffeomorphism. Further, an interesting question is that, under what

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condition on F and γ, $w = P[F]$ is quasiconformal. O. Martio was the first to observe such a question. Pavlović in [19] solved this problem for γ being the unit disk. Kalaj solved this problem for γ being a convex Jordan curve of class $C^{1,\alpha}$ in [8] and for Dini's smooth Jordan curve in [11]. Zhu in [21] considered this problem for general convex Jordan curve. For some different approaches in the plane concerning the class of q.c. harmonic mappings we refer to the papers [3,9,13–18,22,23]. Some recent optimal results on the generalization of this class has been done in [1,12,24]. In this paper we focus our attention in some quantitative estimates of quasiconformal constant of a mapping via its trace F mapping the unit circle onto a strictly convex Jordan curve γ. This is done in Theorems 2.3, 2.4 and 2.5. One of main tools in the proof is Lemma 1.2, which makes itself an interesting result.

1.1. Harmonic functions and Poisson integral

The function

$$P(r,t) = \frac{1-r^2}{2\pi(1-2r\cos t + r^2)}, \quad 0 \le r < 1, \quad t \in [0, 2\pi],$$

is called the Poisson kernel. The Poisson integral of a complex-valued function $F \in L^1(\mathbf{T})$ is a complex-valued harmonic function given by

$$w(z) = u(z) + iv(z) = P[F](z) = \int_0^{2\pi} P(r, t-\tau) F(e^{it}) dt, \tag{1.1}$$

where $z = re^{i\tau} \in \mathbf{U}$. Here $\mathbf{U} := \{z \in \mathbf{C} : |z| < 1\}$ and $\mathbf{T} := \{z \in \mathbf{C} : |z| = 1\}$. On the other hand the following claim holds:

Claim 1: *If w is a bounded harmonic function, then there exists a function $F \in L^\infty(\mathbf{T})$, such that $w(z) = P[F](z)$* (see e.g. [2, Theorem 3.13 b), $p = \infty$]).

We refer to the book of Axler, Bourdon and Ramey [2] for good setting of harmonic functions.

The Hilbert transformation of a function $\chi \in L^1(\mathbf{T})$ is defined by the formula

$$\tilde{\chi}(\tau) = H(\chi)(\tau) = -\frac{1}{\pi} \int_{0+}^{\pi} \frac{\chi(\tau+t) - \chi(\tau-t)}{2\tan(t/2)} \mathrm{d}t.$$

Here $\int_{0^+}^{\pi} \Phi(t)dt := \lim_{\epsilon \to 0^+} \int_\epsilon^\pi \Phi(t)dt$. This integral is improper and converges for a.e. $\tau \in [0, 2\pi]$; this and other facts concerning the operator H used in this paper can be found in the book of Zygmund [25, Chapter VII]. If f is a complex-valued harmonic function then a complex-valued harmonic function $\tilde{f}$ is called the harmonic conjugate of f if $f + i\tilde{f}$ is an analytic function. Notice that such a $\tilde{f}$ is uniquely determined up to an additive constant. Let $\chi, \tilde{\chi} \in L^1(\mathbf{T})$. Then

$$P[\tilde{\chi}] = \widetilde{P[\chi]}, \tag{1.2}$$

where $\widetilde{P[\chi]}$ is the harmonic conjugate of $P[\chi]$ (see e.g. [20, Theorem 6.1.3]).

Assume that $z = x + iy = re^{i\tau} \in \mathbf{U}$. The complex derivatives of a differential mapping $w : \mathbf{U} \to \mathbf{C}$ are defined as follows:

$$w_z = \frac{1}{2}\left(w_x + \frac{1}{i} w_y\right)$$

and