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Universal properties of harmonic functions on trees

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ABSTRACT

We consider an infinite locally finite tree T equipped with nearest neighbor transition coefficients, giving rise to a space of harmonic functions. We show that, except for trivial cases, the generic harmonic function on T has dense range in $\mathbb{C}$. By looking at forward-only transition coefficients, we show that the generic harmonic function induces a boundary martingale that approximates in probability all measurable functions on the boundary of T. We also study algebraic genericity, spaceability and frequent universality of these phenomena.

1. Harmonic functions with dense image

Harmonic functions on infinite trees have been studied in many papers: see [6,14] and their bibliography. At the same time, universal properties have become a subject of growing interest (see [8,4,5] for references). This paper studies harmonic functions on trees from the viewpoint of density properties (Section 1) and universal properties of related boundary martingales (Section 2). A more detailed presentation of Section 2 will appear elsewhere.

We follow most of the terminology established in [1,14]. A *tree* T is a connected, simply connected locally finite infinite graph. We shall also denote by T the set of vertices of the tree. For u, $v \in T$ we write $u \sim v$ if u, v are neighbors. A vertex with only one neighbor is called terminal.

To every pair of vertices of T we assign a non-negative real transition coefficient $p(u, v)$ such that $p(u, v) = 0$ if u and v are not neighbors, in such a way that $\sum_{v:\, v\sim u} p(u, v) = 1$ for all $u \in T$. We shall denote by P the set of these transition coefficients. Actually, in Section 1 we do not need the assumption that the

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transition coefficients are non-negative and add up to 1. We will assume in this Section that $p(u,v)$ is real and it is non-zero if and only if $u \sim v$.

Definition 1. A function $f : T \mapsto \mathbb{C}$ is *P-harmonic* at a non-terminal vertex u if $\sum_{v:\, v\sim u} p(u,v)\, f(v) = f(u)$. f is *P-harmonic on T* if it is P-harmonic at all (non-terminal) vertices.

The space of P-harmonic functions on T is denoted by $H_P(T)$, or $H(T)$. In the remainder of this Section we shall simply call these functions harmonic. The space $H(T)$ is equipped with the relative topology induced by the product (Cartesian) topology of $\mathbb{C}^T$, which is a complete metric topology: the distance between f and $g \in \mathbb{C}^T$ is given by

$$\mathrm{d}(f,g) = \sum_j \frac{1}{2^j} \frac{|f(v_j) - g(v_j)|}{1 + |f(v_j) - g(v_j)|}\,, \tag{1.1}$$

where $\{v_j\}$ is an enumeration of all vertices of T. Clearly, $H(T)$ is closed in $\mathbb{C}^T$, and, by absolute convergence of the series, this distance induces a topology that does not depend on the enumeration.

Note also that, since T is countable, the space of all functions on T (hence also the closed subspace $H(T)$) is separable in this metric: the subspace of all finitely supported complex valued functions with rational real and imaginary parts is countable and dense.

Although analysis on trees is customarily limited to the locally finite case, the results in this paper hold for trees with at most countably many neighbors, by replacing finite sums with series.

In this paper we give some answers to questions of the following general type: how many harmonic functions on T have dense image, and how wild can be the radial behavior at infinity of the martingale associated to a harmonic function?

This beginning section outlines some simple results. These results prove the existence of dense G_δ sets of harmonic functions whose values on certain infinite sets R of vertices are dense in $\mathbb{C}$, and related properties. Such properties hold for a large class of trees. To explain which trees are these, we define ramified sets, as follows.

Definition 2. A vertex with only two neighbors is called *flat*. A *linear branch* is a chain of contiguous flat vertices (in particular, an infinite linear branch is a sub-tree isomorphic to $\mathbb{N}$ or $\mathbb{Z}$, and in the latter case T is isomorphic to $\mathbb{Z}$). For any u, $v \in T$ there exist a unique $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ and a unique minimal path $[u,v] = \{u = z_0 \sim z_1 \sim \cdots \sim z_n = v\}$ of distinct adjacent vertices. The integer n defines a distance $\mathrm{d}(u,v)$ on T. Two vertices u and v are connected by a linear branch if all the intermediate vertices in the minimal path from u to v are flat. An infinite subset $R \subset T$ is called *ramified* if it is not contained in the union of a finite set and finitely many linear branches.

Our results never hold for any R contained in the simplest tree, a copy of the integers (two integers m, n are adjacent if $|n - m| = 1$). Instead, they hold for ramified sets. It is clear that T contains ramified sets (and is itself ramified) if and only if it contains infinitely many vertices with at least three neighbors. If all vertices of T have at least three neighbors, then every infinite subset of T is ramified.

It is convenient to introduce some terminology. Although all results of this Section are stated in a reference free context, the presentation is simpler if we fix a reference vertex $o \in T$: call it the *origin*. Since a tree that contains only terminal vertices consists of at most two vertices, we can choose o non-terminal. The choice of o induces a partial ordering in T: $u \leqslant v$ if u belongs to the minimal path (that is, geodesic arc) from o to v. If $v \neq o$, its *father* v_- is the unique neighbor of v that belongs to the geodesic arc from o to v. The vertices with the same father are called its *children*. More generally, if $v \leqslant w$, then w is called a *descendant* of v.

The length $|u|$ of $u \in T$ is defined as $|u| = \mathrm{d}(o,u)$. For $k \in \mathbb{N}$ let B_k be the *ball* $\{u \in T : |u| \leqslant k\}$.

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