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Reinsurer's optimal reinsurance strategy with upper and lower premium constraints under distortion risk measures[☆]

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ABSTRACT

Motivated by Cui et al. (2013) and Zheng and Cui (2014), we study in this paper the optimal (from the reinsurer's point of view) reinsurance problem where the risk is measured by distortion risk measures, the premiums are calculated under the distortion premium principle, and both the upper and lower premium constraints are involved. Our objective is to seek for the optimal reinsurance strategy which minimizes the reinsurer's risk measure of its total loss. Suppose an reinsurer is exposed to the risk $f(X)$ that is transferred from an insurer, who faces a total loss X and decides to buy from our reinsurer the reinsurance contract. The reinsurance contract specifies that the reinsurer covers $f(X)$ and the insurer covers $X - f(X)$. In addition, the insurer is obligated to compensate our reinsurer for undertaking the risk by paying the reinsurance premium under the distortion premium principle. We present a direct method for discussing the optimization problem. Based on our method, the optimal (or, suboptimal) reinsurance strategy is sought out.

1. Introduction

Reinsurance is the transfer of risk from a direct insurer to the reinsurer at the expense of reinsurance premium calculated under some premium principle. Two important questions to be answered are: What form of reinsurance strategy is optimal for the reinsurer? And, in what sense or subject to what kind of criterion do we call a reinsurance strategy the optimal one?

The optimal reinsurance contract is typically determined by solving an optimization problem, which could involve either maximization or minimization, depending on the chosen criterion. One of the most classical results is based on the variance minimization model. It is proved that the pure stop-loss reinsurance is the optimal reinsurance strategy in that it produces the smallest variance of the insurer's retained loss among all the strategies with the same pure premium, see, for example Borch [1], Gajek and Zagrodny [2], Kaluszka [3] and Kaas et al. [4]. The second classical results correspond to the utility maximization model, which is attributed to Arrow [5]. It asserts that stop-loss reinsurance maximizes the expected utility of the insurer, provided that the insurer has a concave utility function. Other recent results are devoted to the minimization of a number of ingenious risk measures, see, for example Kaluszka [6–8], Gajek and Zagrodny [9], Promislow and Young [10], Cai and Tan [11], Cai et al. [12], Balbás et al. [13], Cheung [14], Chi and Weng [15], Belles-Sampera et al. [16], Chi and Meng [17] and the references therein.

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More recently, Cui et al. [18] discussed the optimal reinsurance problem with the insurer's risk measured by distortion risk measure, the reinsurance premium calculated under the distortion premium principle, and an upper premium constraint. Explicit solutions of the optimal reinsurance strategy minimizing the insurer's risk measure of its total loss are obtained, by presenting a new method. By enlarging the class of admissible ceded loss functions in Cui et al. [18], the optimal reinsurance problem with an upper premium constraint is re-discussed in Zheng and Cui [19], this time the premiums are calculated through the expected value premium principle and the risk is measured by the distortion risk measure. By introducing a premium-adjustment function, the optimal reinsurance treaty minimizing the distortion risk value of the insurer's total liability is also obtained.

However, most of the existing literature focuses on finding optimal reinsurance strategy on behalf of the insurer, results devoting to seeking for optimal reinsurance contract for the reinsurer are still very few. Following Cui et al. [18] and Zheng and Cui [19], the present paper is devoted to finding, among an enlarged admissible ceded loss function class, the optimal reinsurance strategy (contract) that minimizes the distortion risk measure value of the reinsurer's total risk exposure, when premiums are calculated through distortion premium principle and both upper and lower premium constraints are involved. As far as the authors know, the upper premium constraint representing the largest amount the insurer would like to pay for the reinsurance premium is present in many existing results like Gajek and Zagrodny [2,9], Balbás et al. [13], Zhou et al. [20], Sung et al. [21], Tan et al. [22], Cui et al. [18], Zheng and Cui [19] and so on, while the lower premium constraint is absent from the literature. However, adding the lower premium constraint in may be meaningful because the reinsurer is concerned with not only risk management but also profits in practice, i.e., the lower premium constraint is the least profit amount that is acceptable for the reinsurer. It is worth mentioning that the family of distortion risk measures and risk premiums are large and contain very important particular cases such as the family of the comonotone subadditive law invariant coherent risk measures. It needs also to be mentioned that our admissible ceded loss function class is the comparatively large class of all increasing left continuous functions.

The rest of this paper is organized as follows. In Section 2, we present the mathematical description of our optimization problem. Section 3 is devoted to investigating the optimal reinsurance problem with upper and lower premium constraints under distortion risk measure and premium principle. The optimal (or, suboptimal) reinsurance contract on behalf of the reinsurer is sought out.

2. The mathematical presentation of the optimization problem

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the set of all random variables $\mathcal{X}$ defined on this space. Any risk measure ρ is a mapping from a subset of random variables $\mathcal{X}$ to the real line $\mathbb{R}$, $\rho : X \mapsto \rho(X) \in \mathbb{R}$.

Definition 2.1. By Dhaene et al. [23], a function g is called a distortion function if g: $[0, 1] \to [0, 1]$ is a nondecreasing function such that $g(0) = 0, g(1) = 1$. ■

Definition 2.2. Given a distortion function g, functional $\rho_g : \mathcal{X} \to \mathbb{R}$ defined by

$$\rho_g(X) \hat{=} \int_{-\infty}^{0} [g(S_X(x)) - 1]dx + \int_{0}^{\infty} g(S_X(x))dx,$$

is called a distortion risk measure. Hence by definition, the distortion risk measure can be viewed as a(n) (asymmetric) Choquet integral with respect to the set function $g(\mathbb{P})(\cdot)$. ■

In fact, many popular risk measures can be viewed as particular case of distortion risk measure. For example, (i) when the distortion function g is specified to be $\kappa_\alpha(x) = \mathbf{I}_{[\alpha,1]}(x)$, the corresponding distortion risk measure is the Value-at-Risk risk measure $VaR_X(\alpha)$ at level α of a loss X, i.e., $VaR_\alpha(X) \hat{=} S_X^{-1}(\alpha) \hat{=} \inf\{x \in \mathbb{R} \mid S_X(x) \le \alpha\}$, where $S_X(x) \hat{=} \mathbb{P}(X > x)$ is the survival function of X with respect to probability measure $\mathbb{P}$. The Value-at-risk (VaR) has been adopted as a standard tool for assessing the risk and calculating capital requirements in the financial industry. However, there are two disadvantages when using VaR in the financial context. One is that the capital requirements for catastrophic losses based on the measure can be underestimated, i.e., the necessary reserves in adverse scenarios may well be less than they should be. A second drawback is that the VaR may fail the subadditivity property, which is a natural and mild property and allows to decentralize the task of managing the risk arising from a collection of different positions: If separate risk limits are given to different "desks", then the risk of the aggregate position is bounded by the sum of the individual risk limits. (ii) When the distortion function g is specified to be the following function $\gamma_\alpha(x) = \frac{x}{\alpha}\mathbf{I}_{[0,\alpha]}(x) + \mathbf{I}_{(\alpha,1]}(x)$, the corresponding distortion risk measure is the Conditional value-at-risk ($CVaR$), which may be interpreted as the mathematical expectation beyond VaR and is defined as $CVaR_\alpha(X) \hat{=} \frac{1}{\alpha}\int_0^\alpha VaR_q(X)dq = \frac{1}{\alpha}\int_0^\alpha S_X^{-1}(q)dq$. The $CVaR$ risk measure does not suffer the two drawbacks of VaR. However, $CVaR$ has not been widely accepted by practitioners in the financial and insurance industry. (iii) when the distortion function g is specified to be $g_{\alpha,\beta}^{r_1,r_2}(x) \hat{=} \frac{r_1}{\alpha} x \mathbf{I}_{[0,\alpha]}(x) + [r_1 + \frac{r_2 - r_1}{\beta - \alpha}(x - \alpha)]\mathbf{I}_{(\alpha,\beta]}(x) + \mathbf{I}_{(\beta,1]}(x)$, with $\alpha, \beta \in [0, 1]$, $\alpha < \beta$ and $r_1, r_2 \in [0, 1]$, $r_1 \le r_2$, then the corresponding distortion risk measure is the $GlueVaR$ distortion risk measure, which is denoted by $GlueVaR_{\alpha,\beta}^{r_1,r_2}(X)$ and is firstly proposed by Belles-Sampera et al. [16]. It is due to the motivation of providing a risk assessment that lies somewhere between that offered by the VaR and the $CVaR$, that Belles-Sampera et al. [16] propose the