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Bid-ask dynamic pricing in financial markets with transaction costs and liquidity risk

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ABSTRACT

According to the classic no arbitrage theory of asset pricing, in a frictionless market a No Free Lunch dynamic price process associated with any essentially bounded asset is a martingale under an equivalent probability measure. However, real financial markets are not frictionless. We introduce an axiomatic approach of Time Consistent Pricing Procedure (TCPP), in a model free setting, to assign to every financial position a dynamic ask (resp. bid) price process. Taking into account both transaction costs and liquidity risk this leads to the convexity (resp. concavity) of the ask (resp. bid) price. We prove that the No Free Lunch condition for a TCPP is equivalent to the existence of an equivalent probability measure R that transforms a process between the bid price process and the ask price process of every financial instrument into a martingale. Furthermore we prove that the ask (resp. bid) price process associated with every financial instrument is then a R super-martingale (resp. R sub-martingale) which has a càdlàg version.

The axiomatic of TCPP allows for the construction of pricing procedures extending the dynamics of reference assets and calibrated on option prices for a reference family of options. We characterize such TCPP in terms of their dual representation. Such TCPP provide new bounds compatible with the observed bid and ask prices for the reference options and reducing the bid ask spreads for the other financial instruments.

1. Introduction

The theory of asset pricing, and the fundamental theorem, have been formalized by Harrison and Kreps (1979); Harrison and Pliska (1981); Kreps (1981); Dalang et al. (1990); Delbaen and Schachermayer (1994) according to no arbitrage principle. In the usual setting, the market is assumed to be frictionless and then a no arbitrage dynamic price process associated with any essentially bounded asset is a martingale under an equivalent probability measure. However, real financial markets are not frictionless and therefore extensive literature on pricing under transaction costs has been developed. Pioneer works in financial markets with transaction costs are those of Jouini and Kallal (1995) and Cvitanic and Karatzas (1996). A matrix formalism introduced by Kabanov (1999), for pricing under proportional transaction costs has been developed in many papers, as, e.g., in Kabanov and Stricker (2001) or Schachermeyer (2004). In these papers the bid ask spreads are explained by transaction costs. On the contrary the approach developed by Jouini and Kallal (1995) and extended in Jouini (2000) is an axiomatic approach in continuous time assigning to financial assets a dynamic ask price process (resp. a dynamic bid price process) in such a way that the ask price procedure is sublinear.

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In recent years, a pricing theory has also been developed, taking inspiration from the theory of risk measures. This has first been done in a static setting, as in Carr et al. (2001); Föllmer and Schied (2002); Staum (2004); Barrieu and El Karoui (2005) and Bion-Nadal (2005). The axiomatization of dynamic pricing procedures, in the particular context of a Brownian filtration, goes back to Peng (2004) where it is called "g-expectations". A similar notion of monetary utility functions for processes in a discrete time setting has been studied by Cheridito et al. (2006a). A close definition, under the name of Monetary Concave Utility Functional, can be found in Klöppel and Schweizer (2007). Pricing via dynamic risk measures has also been studied by Jobert and Rogers (2005) in discrete time over a finite probability space. We refer to Delbaen (2006) for the study of coherent (or homogeneous) dynamic risk measures, to Detlefsen and Scandolo (2005) and Föllmer and Penner (2006) for dynamic risk measures in a discrete time setting, and to Bion-Nadal (2008, 2009) for dynamic risk measures in a continuous time setting.

Another approach called good deal bounds pricing has been introduced simultaneously by Cochrane and Saa Requejo (2000) and Bernardo and Ledoit (2000). The idea is to exclude some payoffs which are too attractive, known as good deals, in order to restrict the bid ask spread. This corresponds to restricting the set of equivalent martingale measures that one can use to price. Therefore, it is related to the theory of pricing via risk measures. The link, in static case, has been first established by Jaschke and Küchler (2001). For papers along these lines we refer to Cherny (2003), Staum (2004), Bjork and Slinko (2006) and Klöppel and Schweizer (2007b).

In real financial markets, market participants submit offers to buy or sell a fixed number of shares. Therefore, one can observe the resulting limit order book for every traded security. Bid and ask prices associated with nX, where n is a positive integer, can be deduced from the limit order book of X. The resulting ask price of nX is a convex (non-linear) function of the number of shares n. This corresponds to the liquidity risk. Cetin et al. (2004, 2006), and Astic and Touzi (2007), have developed a pricing approach taking into account the liquidity risk.

In this paper we develop an approach in the same vein as Jouini and Kallal (1995), based on the construction of both bid and ask price processes in a model free setting. However, the question addressed here is to define a global dynamic pricing procedure in order to assign to every financial instrument (asset, option, basket, portfolio, etc.) a dynamic bid (resp. ask) price process or even a dynamic limit order book, taking into account both transaction costs and liquidity risk. Assuming that there is at least one liquid asset which is positive, one can take it as numéraire. Furthermore taking into account the lack of liquidity, the usefulness of diversification, the existence of transaction costs, it is natural to make the ask price convex and translation invariant. Another property that we require for a dynamic pricing procedure is the time consistency which means that the price at one instant of time of a financial instrument can be computed indifferently directly or in two steps using an intermediate instant of time. We can then take advantage of the theory of time consistent dynamic risk measures, in particular the existence of a dual representation in terms of probability measures and minimal penalties (Detlefsen and Scandolo, 2005; Bion-Nadal, 2009 for its general form) and the characterization of the time consistency in terms of a cocycle condition for the minimal penalty, (Bion-Nadal, 2008, 2009). Different approaches for pricing already studied in the literature can be formulated in our setting. This is the case in particular for "indifference pricing using exponential utility", for "pricing under portfolio constraints", and for "No good deal pricing".

We define two notions of No Free Lunch for a TCPP, a static one and a dynamic one. Thanks to the time consistency property, we prove that these two notions are equivalent. Our first main result, generalizing that of Jouini and Kallal (1995), states that the No Free Lunch property is equivalent to the existence of (at least) an equivalent probability measure R which transforms a process between the bid process and the ask process of every financial instrument into a martingale. This probability measure R is independent of the financial instrument considered. Our second main result, states that, the dynamic ask (resp. bid) price process associated with any financial instrument is then a R super-martingale (resp. sub-martingale) admitting a càdlàg version, as soon as the TCPP has No Free Lunch. This axiomatic of TCPP also allows for the construction of a dynamic way of pricing extending the dynamics of reference assets $(S^k)_{0\leq k\leq p}$ and compatible with observed bid and ask prices for options $(Y^l)_{1\leq l\leq p}$. The framework is the following: one assumes that one knows the dynamics of some basic assets $(S^k)_{0\leq k\leq d}$ and that for some reference options $(Y^l)_{1\leq l\leq p}$, of maturity date τ_l, a bid and an ask price are observed in the market at one instant in time. Our third main result, is that a TCPP extends the dynamics of the reference assets $(S^k)_{0\leq k\leq d}$ if and only if its dual representation is expressed in terms of local martingale measures for the process $(S^k)_{0\leq k\leq p}$. Furthermore calibrating on option prices induces threshold conditions on the minimal penalties. Our fourth result is that calibrating on option prices leads to reduced spreads compatible with the observed spread for every one of the reference options.

2. Bid-ask dynamic pricing procedure

In this section we introduce the economic model and the definition of Time Consistent Dynamic Pricing Procedure (TCPP).

2.1. The economic model

Throughout this paper one works with a filtered probability space denoted $(\Omega, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, P)$. The filtration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ satisfies the usual assumptions of right continuity and completeness and $\mathcal{F}_0$ is assumed to be the σ-algebra generated by the P null sets of $\mathcal{F}_\infty$. One assumes that the time horizon is infinite, which is the most general case. Indeed if the time horizon is finite equal to T one defines $\mathcal{F}_s = \mathcal{F}_T$ for every $s \geq T$. This general setting covers also the discrete time case.