Nonlinear Analysis: Real World Applications 41 (2018) 401–411

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Normal stability of autonomous and periodic linear Hamiltonian systems

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A R T I C L E I N F O

Article history:
Received 4 October 2016
Received in revised form 25 September 2017
Accepted 31 October 2017

Keywords:
Hamiltonian system
Strong stability
Normal form
Resonances
Normal stability
Stability

A B S T R A C T

We provide necessary and sufficient conditions for normal stability of a linear Hamiltonian system with n degrees of freedom. We also formulate some conditions for strong stability, which show similarities between the concepts of normal stability and strong stability. In the last section, we make the necessary adaptations to extend the concept of normal stability and the results of this paper to periodic Hamiltonian systems.

1. Introduction

We consider initially an autonomous quadratic real function $H = H(\mathbf{q}, \mathbf{p})$ where $(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^{2n}$, $\mathbf{q} = (q_1, \dots, q_n)$, $\mathbf{p} = (p_1 \dots, p_n)$. Let S be the symmetric matrix such that

$$H(\mathbf{z}) = \frac{1}{2}\mathbf{z}^T S\mathbf{z}, \quad \mathbf{z} = (\mathbf{q}, \mathbf{p})^T,$$

and $A = JS$ is a $2n \times 2n$ real Hamiltonian matrix, where J is the standard canonical matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{1}$$

0 is the $n \times n$ null matrix and I is the $n \times n$ identity matrix. The linear Hamiltonian system corresponding to the Hamiltonian function H is given by

$$\dot{\mathbf{z}} = A\mathbf{z}. \tag{2}$$

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https://doi.org/10.1016/j.nonrwa.2017.10.019

Assume that the linear Hamiltonian system (2) is stable, that is, all eigenvalues of A are pure-imaginary and A is diagonalizable over $\mathbb{C}$. We can assume, without loss of generality (see [1,2] for more details), that a linear canonical transformation has already been constructed such that

$$H = \frac{\omega_1}{2}(q_1^2 + p_1^2) + \cdots + \frac{\omega_n}{2}(q_n^2 + p_n^2), \tag{3}$$

where $\pm\omega_1 i, \ldots, \pm\omega_n i$ are the eigenvalues of the linearized system.

To each stable linear Hamiltonian system we can associate a $\mathbb{Z}$-module

$$M_\omega = \{\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n; \quad \mathbf{m} \cdot \boldsymbol{\omega} = m_1\omega_1 + \cdots + m_n\omega_n = 0\}, \tag{4}$$

where $\omega = (\omega_1, \ldots, \omega_n)$ is determined by the normalization process that changes the Hamiltonian function into the form given by (3). If $M_\omega = \{0\}$ we say that the Hamiltonian system (2) does not possess resonances. In the opposite case, the system possesses resonances.

It is a natural conjecture that the properties of the stable linear Hamiltonian system can be expressed by properties of the $\mathbb{Z}$-modules M_ω or other algebraic conditions. In the paper [3], the authors introduce a new concept of stability for autonomous Hamiltonian systems called normal stability and show that a stable linear Hamiltonian is normally stable if and only if the *Moser–Weinstein condition* holds. In this paper, we show other necessary and sufficient conditions for normal stability in the autonomous case, proving Theorem 2.1. Furthermore, we extend this concept to the periodic case and show some results on this new type of stability. In Section 2, we show some necessary and sufficient conditions for strong stability and show a parallel comprehension between this concept and normal stability.

We observe in Section 3 that Theorem 2.1 implies that if the linearized Hamiltonian system associated to an analytic Hamiltonian function H is normally stable then the equilibrium solution of the Hamiltonian system associated to H is Lie-stable.

2. Normal stability in the autonomous case

Consider the $\mathbb{Z}$-module associated to the linear Hamiltonian system (2). Since M_ω is a submodule of the finitely generated module $\mathbb{Z}^n$ and $\mathbb{Z}$ is a principal ideal domain, we have that M_ω is finitely generated, that is, there exist $\mathbf{m}_1, \ldots, \mathbf{m}_s \in M_\omega$ such that

$$M_\omega = \mathbf{m}_1\mathbb{Z} \oplus \cdots \oplus \mathbf{m}_s\mathbb{Z} = \{j_1\mathbf{m}_1 + \cdots + j_s\mathbf{m}_s; \quad j_1, \ldots, j_s \in \mathbb{Z}; \quad \mathbf{m}_1, \ldots, \mathbf{m}_s \in M_\omega\}. \tag{5}$$

In this work we assume that the set of generators $\{\mathbf{m}_1, \ldots, \mathbf{m}_s\}$ of M_ω is minimal, therefore linearly independent and $s < n$. The natural number s is called rank of M_ω and denoted by $s = rank(M_\omega)$.

Suppose that the eigenvalues of A was grouped as follows:

$$\begin{array}{l} \pm k_{11}\omega_1 i, \ \ \pm k_{12}\omega_1 i, \ldots, \pm k_{1s_1}\omega_1 i \\ \pm k_{21}\omega_2 i, \ \ \pm k_{22}\omega_2 i, \ldots, \pm k_{2s_2}\omega_2 i \\ \vdots \ \pm k_{r1}\omega_r i, \ \ \pm k_{r2}\omega_r i, \ldots, \pm k_{rs_r}\omega_r i \end{array} \tag{6}$$

where $\omega_1, \ldots, \omega_r$ are linearly independent over $\mathbb{Q}$.

Define $E(\lambda) = kernel(A - \lambda I) = \{\mathbf{z} \in \mathbb{C}^n; \quad A\mathbf{z} = \lambda\mathbf{z}\}$ to be the complex eigenspace corresponding to an eigenvalue λ. Let

$$W_j = [E(\omega_j k_{j1} i)] \oplus [E(-\omega_j k_{j1} i)] \oplus \cdots \oplus [E(\omega_j k_{js_j} i)] \oplus [E(-\omega_j k_{js_j} i)].$$

Note that W_j satisfies the reality condition that $\lambda \in W_j$ if and only if $\overline{\lambda} \in W_j$ so it is the complexification of a real A-invariant symplectic subspace V_j and

$$\mathbb{R}^{2n} = V_1 \oplus V_2 \oplus \cdots \oplus V_r.$$

Let A_j be the restriction of A to the subspace V_j and H^j be the restriction of H to V_j.