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Structural Stability and Asymptotic Stability for Linear Multidimensional Systems: a Counterexample ⋆

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Abstract: In this paper, we revisit the notions of structural stability and asymptotic stability that are often considered as equivalent in the field of multidimensional systems. We illustrate that the equivalence between asymptotic and structural stability depends on where we define the boundary conditions. More precisely, we show that structural stability implies asymptotic stability when the boundary conditions are imposed on the positive axes. But a carefully designed counterexample shows that the opposite does not hold in this case. This illustrates once again the importance of the boundary conditions when dealing with multidimensional systems.

Keywords: asymptotic stability, structural stability, multidimensional systems, linear systems, discrete systems

1. INTRODUCTION

The concept of stability is probably one of the most natural ideas in the field of control systems. It can take multiple forms depending on the problem analyzed and countless definitions have been proposed in the literature: asymptotic, exponential, finite-time, input-output, etc. These definitions have all been extended to different types of models, including multidimensional systems *i.e.* models where the information propagates along two or more independent directions. For readers interested in multidimensional systems, also called nD models, one can refer to the following contributions Kaczorek (1985); Zerz (1998); Rogers et al. (2007); Bose (2010).

In this paper, we are interested in the relations between structural stability and asymptotic stability of two dimensional systems when we work with a Fornasini-Marchesini model [1] (Fornasini and Marchesini (1976, 1978)). Contrary to the 1D case, these two notions of stability can slightly vary in the literature. For instance, Fornasini and Marchesini (1978) reduces asymptotic stability to attractivity (trajectories converging to the equilibrium point) whereas Liu and Michel (1994); Yeganefar et al. (2013b) consider asymptotic stability as the sum of the classical concepts of stability and attractivity. Similarly, the notion of structural stability is sometimes introduced either as a criterion or as a definition (Li et al. (2013); Bachelier et al. (2016)) and not always called structural stability (Valcher (2000); Scheicher and Oberst (2008); Oberst and Scheicher (2014)). It has also been extended to various setups such as continuous or mixed continuous-discrete models (Chesi and Middleton (2014)).

With these warnings in mind, since Fornasini and Marchesini (1978), it is known that structural stability and asymptotic stability are equivalent if we consider a discrete Fornasini-Marchesini model with a special choice of boundary conditions. We also know, since Valcher (2000); Yeganefar et al. (2013b), that the choice of boundary conditions is crucial to whatever concept of stability we decide to work with. However it is common in the literature to find claims where structural and asymptotic stability are considered equivalent even if the model and the boundary conditions do not fit into the framework introduced in Fornasini and Marchesini (1978). In this paper, we show that if we decide to work with a "natural" choice of boundary conditions that we will explicit in Section 2, then structural stability implies asymptotic stability but the opposite does not hold. To highlight this last point, we will present a counterexample of a system asymptotically stable but not structurally stable.

The paper will therefore be organized as follows. The next section will clarify the concepts of structural and asymptotic stability, recall previous results relating these two concepts and highlight the choice of the boundary con-

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[1] The second widely used model, called the Roesser model (Roesser (1975)), will not be analyzed in this paper.

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ditions. Section 3 will show that structural stability implies asymptotic stability according to the definitions given in the Section 2. Finally in Section 4, we will introduce a simple discrete Fornasini model. We will show that this system is asymptotically stable but structurally unstable. This implies that both concepts are not equivalent or, as some authors claim, structural stability is not a necessary and sufficient condition for asymptotic stability. A conclusion will highlight once again the contributions of the paper and discuss the future questions we would like to answer.

Notations

- $\mathbb{R}^n$ is the vector space of dimension $n \in \mathbb{N}^*$ over the field $\mathbb{R}$. It will be endowed with some norm $\|\cdot\|$. The space of square real matrices $\mathbb{R}^{n\times n}$ of dimension n is endowed with the induced matrix norm, still denoted by $\|\cdot\|$.
- $(\mathbb{R}^n)^{\mathbb{N}}$ is the space of $\mathbb{R}^n$-valued sequences of one index. We will write $c_0(\mathbb{R}^n)$ its subset of sequences converging to zero at infinity. We endow this subset with the infinity norm $\|\cdot\|_\infty$ corresponding to the supremum of all the modulus of the elements of the sequence.
- $[\cdot\cdot]$ stands for the floor function (the greatest preceding integer of a real number).
- I_n is the identity matrix.
- The big O notation will be noted $\mathcal{O}$ and the symbol for the asymptotic equivalence will be $\sim$.

2. DEFINITION OF STRUCTURAL AND ASYMPTOTIC STABILITY

2.1 The model and the choice of the boundary conditions

A 2D discrete Fornasini-Marchesini second model is defined as follows:

$$x(i+1, j+1) = Ax(i, j+1) + Bx(i+1, j) \qquad (1)$$

where x is the state vector of dimension n, $A, B \in \mathbb{R}^{n\times n}$ are non-zero matrices, i and j are two indexes that are usually taken either in $\mathbb{Z}$ or $\mathbb{N}$.

Notice that in order to solve this equation one needs to impose boundary conditions on a sufficiently large set contrary to the 1D case ($x(i+1) = Ax(i)$) where the condition is reduced to a vector. The choice of the set defining the boundary conditions is not unique. Several authors, for reasons exposed in Fornasini and Marchesini (1978) have chosen to impose the boundary conditions on the line $\left\{(i, j) \in \mathbb{Z}^2 \;/\; i + j = 0\right\}$ therefore working with indexes in a subset of $\mathbb{Z}^2$.

This is however not a natural choice to make for instance in terms of computation. It seems much more natural to choose the boundary conditions on the first quadrant, *i.e.* by imposing the choice of $x(0, j)$ and $x(i, 0)$. In this case, we will note : $\forall (i, j) \in \mathbb{N}^2,\ \Psi(i, j) := (\Psi_1(j), \Psi_2(i))$, where $\Psi_1(j) := x(0, j)$ and $\Psi_2(i) := x(i, 0)$.

2.2 Structural stability and asymptotic stability

As pointed earlier, structural stability is sometimes used as a criterion in the literature and sometimes introduced as a definition, see e.g. Fornasini and Marchesini (1978); Li et al. (2013) and Bachelier et al. (2016).

Definition 1. A Fornasini model (1) is said to be *structurally stable* if,

$$\det\big(I_n - \lambda A - \mu B\big) \neq 0, \forall\, \lambda, \mu \in \mathbb{C},\ |\lambda| \leq 1,\ |\mu| \leq 1. \quad (2)$$

Remark 2. For one-dimensional systems the corresponding definition is $\det\big(I_n - \lambda A\big) \neq 0$ for all $\lambda \in \mathbb{C}$, with $|\lambda| \leq 1$, which means that all the eigenvalues of A belongs to the open unit ball. This is a well-known necessary and sufficient condition for asymptotic stability (Hahn, 1967, Chapter 2).

In the literature, asymptotic stability has been defined in slightly different manners depending on the type of system (linear or non-linear) and the imposed boundary conditions. We here remind the reader of the two main definitions that will be considered in this paper.

Definition 3. (Fornasini and Marchesini (1978)). The system (1) with bounded boundary conditions on the line $\left\{(i, j) \in \mathbb{Z}^2 \;/\; i + j = 0\right\}$ is said to be *asymptotically stable* if

$$\lim_{r\to\infty} \sup_{n\in\mathbb{Z}} \|x(r-n, n)\| = 0.$$

We now redefine the previous notion when the boundary conditions are imposed on $\mathbb{N}^2$.

Definition 4. (Yeganefar et al. (2013b)). The system (1) with $(i, j) \in \mathbb{N}^2$ and boundary conditions $\Psi_1(j)$ and $\Psi_2(i)$ is said to be *asymptotically stable* if

(1) $\forall \epsilon > 0$ there is a $\delta_\epsilon > 0$ such that if $\|\Psi\|_\infty = \max\big(\|\Psi_1(j)\|, \|\Psi_2(i)\|\big) < \delta_\epsilon$ then $\|x(i, j)\| < \epsilon$ for all $(i, j) \in \mathbb{N}^2$.
(2) $\lim_{i+j\to\infty} x(i, j) = 0$ when $\lim_{j\to\infty} \Psi_1(j) = \lim_{i\to\infty} \Psi_2(i) = 0$.

Remark 5. Note that if the indexes are taken in $\mathbb{N}^2$, the condition $\lim_{r\to\infty} \sup_{n\in\mathbb{Z}} \|x(r-n, n)\| = 0$ is equivalent to $\lim_{i+j\to\infty} x(i, j) = 0$.

Finally, we also need to introduce the concept of exponential stability that will be used in the next section.

Definition 6. (Yeganefar et al. (2013b)). The system (1) is said to be *exponentially stable* if there are constants $q \in (0, 1)$ and $M > 0$ such that for any boundary conditions Ψ_1 and Ψ_2, and for all $(i, j) \in \mathbb{N}^2$, we have:

$$\|x(i, j)\| \leq M \left(\sum_{s=1}^{j} \frac{\|\Psi_1(s)\|}{q^{s+1}} + \sum_{r=1}^{i} \frac{\|\Psi_2(r)\|}{q^{r+1}} \right) q^{i+j}. \quad (3)$$

Remark 7. With a change of variable, (3) can also be written as

$$\|x(i+1, j+1)\| \leq M \left(\sum_{k=0}^{j} q^{i+k} \|\Psi_1(j+1-k)\| + \sum_{k=0}^{i} q^{k+j} \|\Psi_2(i+1-k)\| \right). \quad (4)$$

We will not discuss here the reasons behind these definitions, the reader can refer to Yeganefar et al. (2013b). Let us now recall one of the main result of the original paper by Fornasini and Marchesini (1978) that links structural stability and asymptotic stability: