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Incomplete financial markets and contingent claim pricing in a dual expected utility theory framework

M. Corradini, A. Gheno *

University of Rome III, Department of Economics, via Silvio d'Amico 77, 00145 Roma, Italy

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ABSTRACT

This paper investigates the price for contingent claims in a dual expected utility theory framework, the dual price, considering arbitrage-free financial markets. A pricing formula is obtained for contingent claims written on n underlying assets following a general diffusion process. The formula holds in both complete and incomplete markets as well as in constrained markets. An application is also considered assuming a geometric Brownian motion for the underlying assets and the Wang transform as the distortion function.

1. Introduction

The Black and Scholes (1973) and Merton (1973) (BSM) framework introduced in the 1970s is still the most widely used in the industry as far as contingent claim pricing is concerned. Within this approach the absence of arbitrage opportunities allows identification of the price of a financial contingent claim as the price of a self-financing replicating portfolio. In particular, this price can be expressed as the discounted expected value of the contingent claim future pay-off under a unique risk-neutral measure.

Unfortunately, the BSM approach is valid only in a complete and unconstrained financial market, where for any financial contingent claim there exists a self-financing portfolio replicating its pay-off. As can be shown, in incomplete markets the assumption of absence of arbitrage opportunities does not allow selecting a unique equivalent martingale measure to be used as the risk-neutral measure. It results that the no-arbitrage price of a financial contingent claim is not uniquely determined and lies between an upper and a lower bound.[1]

In order to select a risk-neutral measure among the infinitely many equivalent martingale measures or, equivalently, a unique arbitrage-free price of a financial contingent claim, among others a utility based approach has been followed in the literature. For example, using the fair price notion obtained in Davis (1998) as a formalization of the principle of equimarginal utility formulated by Jevons (1970), in Karatzas and Kou (1996) a unique utility based fair price is obtained for a financial contingent claim in the general case of an incomplete [2] and constrained financial market. [3]

In the present paper, in order to select a unique martingale measure among the infinitely many compatible with the absence of arbitrage opportunities, an approach based on the Choquet integral or, from an economics perspective, on Yaari's theory of choices, the so-called dual expected utility (DEU) theory, presented in Yaari (1987), Schmeidler (1986) and Schmeidler (1989), is adopted. As shown by several authors since the 1960s, actual decisions are not fully consistent with the axioms of the classical expected utility (EU) theory presented by von Neumann and Morgenstern (1944) and, as shown in Quiggin (1993), an important advantage in using the DEU theory is that it solves the Allais paradox (Allais, 1953), originating from the interpretation of the choices by the EU theory. Moreover "currently there is a pressing need for the determination of the fair value of financial and insurance risks" (Wang, 2002) and, although it has its own paradoxes, Yaari's theory of choice seems very suitable for this purpose.

In particular, in Wang (2000, 2002) an expression for the risk-adjusted premium for a risk R, $H[R, \alpha]$, is given in terms of

* Corresponding author.
E-mail addresses: m.corradini@uniroma3.it (M. Corradini), gheno@uniroma3.it (A. Gheno).

1 See for instance (Karatzas and Kou, 1996).

2 A market is incomplete if there exists a contingent claim whose pay-off cannot be replicated by a self-financing portfolio.

3 In the presence of constraints on portfolio choice, for example short sales not allowed, the market is said to be constrained.

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a Choquet integral using the Wang transform as the distortion function. In these papers the assumption that the assets can be priced by applying such a risk-adjusted premium to the present value of the future asset price (hereafter called Wang's assumption), allows the derivation of an implied value of the parameter α.[4]

Such an approach has been adopted for European call option pricing and the standard BSM formula has been recovered in a complete and unconstrained arbitrage-free financial market with a risk-free asset and a risky asset following a geometric Brownian motion.

In Hamada and Sherris (2003), the approach developed in Wang (2000, 2002) is formally considered in a complete and unconstrained arbitrage-free financial market with a risk-free asset and a risky asset following a geometric Brownian motion. In particular, using Wang's assumption, a pricing formula for a contingent claim whose pay-off is comonotone [5] with the terminal value of the underlying asset is obtained.

The aim of this paper is to generalize the results of Wang (2000, 2002) and Hamada and Sherris (2003). In fact, in the present paper, (a) n risky assets following a general diffusion process are considered in an incomplete and generally constrained financial market; (b) in this framework a pricing formula for contingent claims whose pay-off is not necessarily comonotone with the terminal value of the underlying assets is deduced without Wang's assumption.

In particular, in this paper, within the DEU theory framework, the price of a financial contingent claim is obtained using the concept of indifference price, i.e. the price $\hat{p}$ of a financial contingent claim chosen by the agent in such a way that it is indifferent for him to buy (a) an optimal portfolio at price x or (b) an optimal portfolio at price $x - \hat{p}$ and a financial contingent claim at price $\hat{p}$. A pricing formula is obtained for a financial claim contingent on n underlying assets following a general diffusion process in an incomplete market with general constraints on portfolio choice, for a generic distortion function describing a risk-averse agent, without any comonotonicity hypothesis between its pay-off and the underlying assets. This pricing formula also holds in a complete market.

A market with n risky assets following geometric Brownian motion and a risk-free asset with constant drift is then considered using the Wang transform as the distortion function. In a complete and unconstrained financial market it is shown that the dual price is equivalent to the well-known BSM price or, in other words, the price of a financial contingent claim is obtained as the present value of its expected pay-off under the unique equivalent martingale measure. This result also holds if the n risky underlying assets are traded in a market where short sales are not allowed, i.e. in a constrained market. If only the first m risky assets are traded and the remaining $n-m$ are not traded, i.e. if the market is incomplete, the selected martingale measure, or in other words the market price of risk for non-traded assets, is explicitly obtained.

Our paper is also related to the work by Goovaerts and Laeven (2008) (see also the companion paper Goovaerts et al. (2004)) who characterize axiomatically pricing principles for contingent claims on risky assets of which the price dynamics are governed by general diffusion processes (including geometric Brownian motion as a specific example). Their financial market is allowed to be incomplete and the derived pricing principles are consistent with arbitrage-free pricing. The pricing principles characterized involve a probability measure transform related to the Wang transform, and is coined the Esscher–Girsanov transform by Goovaerts and Laeven (2008).

The remainder of the paper is structured as follows. Outlines of EU and DEU theories are given in Section 2. In Section 3, the financial market model with general diffusion processes is described. In Section 4, the concept of dual price is introduced, and a pricing formula for a contingent claim is obtained both in complete and incomplete markets. In Section 5, the geometric Brownian motion and the Wang transform are considered as a specific example; the corresponding pricing formulae are obtained. In the final section, the conclusions are drawn.

2. DEU decision theory

The EU theory by von Neumann and Morgenstern (1944) is the most frequently used approach to solve problems of choice under uncertainty. The main disadvantage in using such an approach is that, as shown by several authors since the 1960s, actual decisions are not fully consistent with all EU theory axioms.

Following Quiggin (1993), the EU theory axioms can be expressed as follows.

(A.1) If the random variables X and Y have the same cumulative distribution function then $X \sim Y$.

(A.2) The preference relation $\succeq$ is a weak order. This means that $\succeq$ is complete, transitive and reflexive.

(A.3) If X first stochastically dominates Y then $X \succeq Y$.

(A.4) The preference relation $\succeq$ is continuous.

(A.5) If F, G, H are cumulative distribution functions with F preferred to G, then for any $p \in [0, 1]$ the probability mixture $pF + (1-p)H$ is preferred to $pG + (1-p)H$.

Here $X \succeq Y$ is short for Y not preferred to X and $X \sim Y$ is short for X being indifferent to Y.

As shown, for example, in Allais (1953), A.5, the so-called independence axiom, is violated in several empirical tests. In order to avoid this problem, theories of choice alternative to the EU theory, called non-expected utility theories, have been presented in the literature. In particular, as shown in Quiggin (1993), the DEU theory is a non-expected utility theory whose axioms are A.1, A.2, A.3, A.4; and axiom A.5 is replaced with

(A.5*) If the random variables X, Y, Z are comonotonic and $X \succeq Y$, then, for any $p \in [0, 1]$, $pX + (1-p)Z \succeq pY + (1-p)Z$.

Under axioms A.1, A.2, A.3, A.4 and A.5*, there exists a non-decreasing function $g : [0, 1] \to [0, 1]$, with $g(0) = 0$ and $g(1) = 1$, such that

- $X \succeq Y \Leftrightarrow \mathbb{E}_g[X] \geq \mathbb{E}_g[Y]$;
- $X \preceq Y \Leftrightarrow \mathbb{E}_g[X] \leq \mathbb{E}_g[Y]$;
- $X \sim Y \Leftrightarrow \mathbb{E}_g[X] = \mathbb{E}_g[Y]$,

with

$$\mathbb{E}_g[X] = \int_{-\infty}^{+\infty} x \mathrm{d}g\left(F_X(x)\right),$$

where $F_X(x)$ is the probability distribution function of the random variable X.

Hence in the DEU framework "attitudes toward risks are characterized by a distortion applied to probability distribution functions, in contrast to expected utility in which attitudes toward risks are characterized by a utility function of wealth" (Wang and Young, 1998).

The analytical form of the so-called distortion function, g, embeds the degree of aversion towards risk of the decision maker. In particular, if g is increasing and concave, as shown in Quiggin (1993), the decision maker is risk-averse and the resulting ordering

4 Wang's risk-adjusted premium can be viewed as an application of the principle of equivalent utility; see e.g. Denuit et al. (2006).

5 The random variables X and Y are comonotone if there exist a random variable Z and two not decreasing real functions f and h such that $X = f(Z)$ and $Y = h(Z)$.