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Feedback Stability for Dissipative Switched Systems

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Abstract: A method is proposed to infer Lyapunov and asymptotic stability properties for switching systems, under arbitrary continuous-state feedback. Continuous-time systems which are dissipative in the multiple-storage function sense are considered. A partition of the state space, induced by the cross-supply rates and the feedback function, is used to derive conditions for stability. It is argued that the conditions proposed here are more straightforward to check, when compared to those proposed by other approaches in the literature. Some numerical examples are offered to illustrate this point.

Keywords: Stability analysis; Dissipativity properties; Switching systems; Application of nonlinear analysis and design.

1. INTRODUCTION

Switching systems are dynamic systems, for which the system dynamics switch discretely between different modes of operation or subsystems. Interest in such systems is part of a larger trend, which also includes impulsive systems Haddad et al. (2014), which exhibit discontinuities in the system state, and hybrid systems Goebel et al. (2012), for which switching and impulsive behaviours are combined. The study of these classes of systems has been motivated by the fact that this type of hybrid behaviour is observed in various natural and artificial processes, and the observation that the intricacies of such behaviours could not be captured by pre-existing theories.

Stability is a central problem, both for smooth and for switching systems. Extending the non-switching stability results and techniques (for instance, Lyapunov and Lasalle techniques) to the case of switching systems has been a non-trivial endeavour. Such extensions are explored in Liberzon (2012), where single Lyapunov functions are used, and in Branicky (1998), where multiple Lyapunov functions are applied. Another popular approach includes the introduction of dwell-time conditions: that is, restrictions for the time that the system spends in every mode of operation, as proposed, for example in Shorten et al. (2007). Other problems, usually explored in relation to switching systems, include decidability of various control problems (reachability, controllability etc) Henzinger et al. (1995), and verification Broucke (1999); Navarro-López and Carter (2016). For those problems, systems are represented as hybrid automata, and tools from logics Davoren and Nerode (2000) and reachability theory Lygeros et al. (1999) are adapted to automatically answer questions about the evolution of the trajectories.

Dissipativity was first introduced for continuous-time (non-switching) dynamical systems, in 1972, in Willems (1972). The idea behind it was that the insights gained by the use of the concept of energy, for example in electric circuits, could be replicated in more abstract dynamic systems. Energy descriptions of dynamic systems are desirable, principally for two reasons. First, they allow for intuitively clear descriptions of system behaviours, and, second, they are versatile, in the sense that a diverse collection of phenomena can be described in terms of energy.

Within the dissipativity framework, the behaviour of the system is described by a supply rate function, which describes the flow of energy in and out of the system, and a storage function, expressing the energy stored in the system at every state. A system is said to be dissipative, if, as it evolves, it dissipates some of the energy that flows into it (which is a form of 'mild' behaviour); this idea is made precise in Definition 3 of the next section. For such systems, information about the trajectories can be obtained by studying the energy behaviour.

In the case of switching systems, various extensions of dissipativity theory have been proposed. The majority of those extensions falls within one of two categories. First, single-storage function definitions (Haddad and Chellaboina (2001) and Naldi and Sanfelice (2011)), for which a common supply rate/storage function pair is used to describe the energy behaviour of all the subsystems. Second, multiple-storage function definitions (Zhao and Hill (2008), Zefran et al. (2001), Pogromsky et al. (1998) and Navarro-López and Laila (2013)), for which one pair is used for each subsystem, and the concept of cross-supply rate is introduced to describe the energy effects of the interconnection. Other approaches include the passivity indices, introduced in McCourt and Antsaklis (2010), where switching dissipativity is not defined *per se*, but the dissipativity properties of the subsystems are used to establish results; and the differential inclusions approach proposed in Haddad and Sadikhov (2012), which innovates by using multiple supply rates.

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In this work, the multiple-storage function dissipativity framework is used to establish some stability results for switching systems. Similar results already exist in the literature (see the discussion in Section 3), corresponding to the various multiple-storage function definitions of dissipativity. The proposed approach differs from these results, because it deploys the dissipativity framework in a distinct way, and, hence, produces a characterisation of the stability properties that is substantially different. It is argued that this characterisation is often preferable, in the sense that the conditions that it posits require information that is relatively easy to obtain, in comparison to already-existing methods.

The rest of this paper is organised as follows. First, some preliminaries are given, and the main concepts are formally defined. Then, the main results are presented, along with their proofs, and a comparison with already-existing results. Finally, some illustrative examples are given, followed by brief concluding remarks.

2. PRELIMINARIES

For $n, m_o, m_i \in \mathbb{N}$, some natural numbers, let $X = \mathbb{R}^n$, $U = \mathbb{R}^{m_i}$, $Y = \mathbb{R}^{m_o}$ designate the domains of the system state, the input and the output. Take a finite collection of indices $\mathcal{N} = \{1, 2, \ldots, N\}$, and a corresponding collection of functions $\mathcal{F} = \{f_1, f_2, \ldots, f_N\}$, such that $\forall i \in \mathcal{N}$, $f_i : X \times U \mapsto \mathbb{R}^n$. Take, also, a similar collection of output functions $\mathcal{O} = \{h_1, h_2, \ldots, h_N\}$.

In order to describe the switching behaviour, consider $\sigma : \mathbb{R}^+ \mapsto \mathcal{N}$, a piecewise constant, left-continuous function, representing the switching law of the system. Take, also, $\mathcal{T}$ to be the increasing sequence of switching instants $(t_k)_{k\in\mathbb{N}}$. Then, for some $t_k, t_{k+1} \in \mathcal{T}$ and $t_k < \tau_0 \leq t_{k+1} < \tau_1$, it always holds that $\sigma(t_k + 1) = \sigma(\tau)$, but it might be that $\sigma(t_k + 1) \neq \sigma(\tau_1)$.

The systems that will be considered here take the following form:

$$\mathcal{H} \begin{cases} \dot{x}(t) = f_{\sigma(t)}(x(t), u(t)), \\ y(t) = h_{\sigma(t)}(x(t), u(t)), \end{cases} \tag{1}$$

with $\sigma(t)$ being the switching law. The notation $f_{\sigma(t)}$ means that $f_{\sigma(t)} = f_i$, when $\sigma(t) = i \in \mathcal{N}$; the same convention holds for $h_{\sigma(t)}$. The implication, then, is that, for $\mathcal{H}$, the system dynamics between consecutive switches are given by one of the functions in $\mathcal{F}$. It is common to refer to these functions as the subsystems of $\mathcal{H}$. It is said, then, that the subsystem i is active, when $\sigma(t) = i$. The elements of $\mathcal{F}$ are assumed to be continuous, and, these elements and the input functions u are assumed to be well-behaved, so that existence and uniqueness issues do not arise.

An additional issue, relevant to the study of switching systems, is the well-known Zeno behaviour, in which the solution of the system cannot be extended beyond some time point, due to the presence of infinite switches. It is assumed, here, that the switching regimes under consideration do not exhibit this kind of behaviour.

An equilibrium point for the system $\mathcal{H}$ is a point $x_* \in X$, for which some $f_i \in \mathcal{F}$ vanishes, for some $u_* \in U$. That is:

Definition 1. (Equilibrium point Khalil (2002)). An equilibrium point for $\mathcal{H}$ is a triplet $(x_*, u_*, i) \in X \times U \times \mathcal{N}$, such that $f_i(x_*, u_*) = 0$. Let $\mathcal{E}$ designate the set (possibly empty) of equilibrium points of $\mathcal{H}$. ■

Observe that, for every $i \in \mathcal{N}$, multiple equilibrium points might exist.

In this work, some stability properties of the equilibria of switching systems will be examined. The following notion of stability is used.

Definition 2. (Stability Zhao and Hill (2008)). Consider a system $\mathcal{H}$, starting at $t_0 \geq 0$, with initial state $x_0 = x(t_0) \in X$ under some control $u(t)$ and some switching rule $\sigma(t)$. An equilibrium point $e = (x_*, u_*, i) \in \mathcal{E}$ of $\mathcal{H}$, under the some control $u(t)$, is said to be:

- **attractive**, iff $\lim_{t\to\infty} x(t) = x_*$.
- **Lyapunov stable**, iff for each $\epsilon > 0$, there exists $\delta > 0$, such that, if $\|x_* - x(t_0)\| < \delta$, then $\|x_* - x(t)\| < \epsilon$, for all $t \geq t_0$.
- **asymptotically stable**, if it is both attractive and stable. ■

In the next section, some stability conditions will be derived for the subset of the switching systems that are dissipative. To that effect, a multiple-storage function definition of dissipativity, introduced in Zhao and Hill (2008), is used.

Definition 3. (Dissipativity Zhao and Hill (2008)). A system $\mathcal{H}$ is said to be dissipative, with respect to a collection of supply rates $\{s_i\}_{i\in\mathcal{N}}$, and a collection of cross-supply rates $\{\{r_{ij}\}_{j\in\mathcal{N}/i}\}_{i\in\mathcal{N}}$ ($/$ is used to denote the relative complement) , where $\forall i, j \in \mathcal{N}$, $s_i : U \times Y \mapsto \mathbb{R}$, and $r_{ij} : X \times U \times Y \mapsto \mathbb{R}$, with all s_i, r_{ij} locally integrable, if there exists a collection of functions $\{V_i\}_{i\in\mathcal{N}}$, with $V_i : X \mapsto \mathbb{R}^+$, called the storage functions, such that, $\forall t_k, t_{k+1} \in \mathcal{T}$, when $t_k \leq t_k^1 \leq t_k^2 < t_{k+1}$:

(1) $V_i(x(t_k^1)) - V_i(x(t_k^2)) \leq \int_{t_k^1}^{t_k^2} s_i(u(t), y(t))dt$, if $\sigma(t_k) = i$.

(2) $V_i(x(t_k^1)) - V_i(x(t_k^2)) \leq \int_{t_k^1}^{t_k^2} r_{ij}(x(t), u(t), y(t))dt$, if $\sigma(t_k) = j \neq i$.

For the level lines of V_i, the notation $N_i(\epsilon) = \{x \in X \mid V_i(x) \leq \epsilon\}$ is used.

In essence, the definition posits that a system is dissipative when all its component systems (that is, the members of $\mathcal{F}$) are dissipative (with respect to some arbitrary supply rates), and it introduces the concept of the cross-supply rate, in order to capture the transfer of energy to some component, caused by the activity of some other component.

In Definition 3, the supply and cross-supply rates satisfy almost identical inequalities and express energy transfers; they appear, then, to be conceptually similar. There is, however, an important distinction which has to be made between them. A supply rate for some subsystem expresses a property of that subsystem, namely, a deep relation between its inputs, its outputs and its state. A cross-supply rate, on the other hand, is an artifact of the connection: that is, of the fact that two subsystems are components of some switching system. Therefore, while the former is