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A stability criterion for non-degenerate equilibrium states of completely integrable systems

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Abstract

We provide a criterion in order to decide the stability of non-degenerate equilibrium states of completely integrable systems. More precisely, given a Hamilton–Poisson realization of a completely integrable system generated by a smooth n-dimensional vector field, X, and a non-degenerate regular (in the Poisson sense) equilibrium state, $\overline{x}_e$, we define a scalar quantity, $\mathcal{I}_X(\overline{x}_e)$, whose sign determines the stability of the equilibrium. Moreover, if $\mathcal{I}_X(\overline{x}_e) > 0$, then around $\overline{x}_e$, there exist one-parameter families of periodic orbits shrinking to $\{\overline{x}_e\}$, whose periods approach $2\pi/\sqrt{\mathcal{I}_X(\overline{x}_e)}$ as the parameter goes to zero. The theoretical results are illustrated in the case of the Rikitake dynamical system.

MSC: 37J25; 37J45; 37J05; 86A25

Keywords: Completely integrable systems; Stability; Equilibrium states; Periodic orbits; Rikitake system

1. Introduction

The aim of this article is to provide a criterion in order to decide the stability of non-degenerate equilibrium states of completely integrable systems. More precisely, given a Hamiltonian realization (of Poisson type) of a completely integrable system generated by a smooth n-dimensional vector field, X, and a non-degenerate regular (in the Poisson sense) equilibrium state, $\overline{x}_e$, we define a scalar quantity, $\mathcal{I}_X(\overline{x}_e)$, whose sign determines the stability of $\overline{x}_e$,

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i.e., *if* $\mathcal{I}_X(\overline{x}_e) > 0$ *then* $\overline{x}_e$ *is Lyapunov stable, whereas if* $\mathcal{I}_X(\overline{x}_e) < 0$ *then* $\overline{x}_e$ *is unstable*. Moreover, as the characteristic polynomial of the linearization of X at $\overline{x}_e$, $\mathfrak{L}^X(\overline{x}_e)$, is given by $p_{\mathfrak{L}^X(\overline{x}_e)}(\mu) = (-\mu)^{n-2} \cdot \left(\mu^2 + \mathcal{I}_X(\overline{x}_e)\right)$, it follows that $\mathcal{I}_X(\overline{x}_e)$ depends only on X and $\overline{x}_e$, and not on the Hamiltonian realization. Also, if we denote by $\Sigma_{\overline{x}_e}$, the symplectic leaf (passing through $\overline{x}_e$) of the Poisson configuration manifold of the Hamiltonian realization, then the sign of $\mathcal{I}_X(\overline{x}_e)$ determines again the stability of $\overline{x}_e$, this time regarded as an equilibrium state of the restricted vector field $X|_{\Sigma_{\overline{x}_e}}$. Moreover, if $\mathcal{I}_X(\overline{x}_e) > 0$, then there exists $\varepsilon_0 > 0$ and a one-parameter family of periodic orbits of $X|_{\Sigma_{\overline{x}_e}}$ (and hence of X too), $\{\gamma_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0} \subset \Sigma_{\overline{x}_e}$, that shrink to $\{\overline{x}_e\}$ as $\varepsilon \to 0$, with periods $T_\varepsilon \to \frac{2\pi}{\sqrt{\mathcal{I}_X(\overline{x}_e)}}$ as $\varepsilon \to 0$. Also, the set $\{\overline{x}_e\} \cup \bigcup_{0<\varepsilon<\varepsilon_0} \gamma_\varepsilon$ represents the connected component of $\Sigma_{\overline{x}_e} \setminus \gamma_{\varepsilon_0}$, which contains the equilibrium point $\overline{x}_e$. Note that by choosing a different Hamiltonian realization of the completely integrable system, for which $\overline{x}_e$ is also a non-degenerate regular equilibrium point, we obtain the existence of a different family of periodic orbits with the same properties, this time the orbits being located on the regular symplectic leaf (passing through $\overline{x}_e$) corresponding to the Poisson configuration manifold associated to this specific Hamiltonian realization. On the applicative level, all theoretical results are illustrated in the case of the Rikitake dynamical system.

More precisely, the structure of the article is the following: the second section presents the geometry associated to a general completely integrable system. More precisely, using the property that any completely integrable system admits Hamiltonian realizations of Poisson type, we illustrate the associated Poisson geometry, and its relations with the dynamics generated by the system. The aim of the third section is to characterize the set of equilibrium states of a general completely integrable system, and also to analyze the geometric and analytic properties of certain subsets of equilibria, naturally associated with the Poisson geometry of the Hamiltonian realizations of the system. In fourth section of the article we define the scalar quantity $\mathcal{I}_X(\overline{x}_e)$, and analyze its main geometric and analytic properties. The fifth section is the main part of this article and contains the main result, which provides a criterion to test the stability of non-degenerate regular equilibrium states of Hamiltonian realizations of completely integrable systems. The aim of the sixth section is to give a criterion to decide leafwise stability of non-degenerate regular equilibria of Hamiltonian realizations of completely integrable systems, and also to study the local existence of periodic orbits. In the last section, we illustrate the main theoretical results in the case of a concrete example coming from geophysics, namely, the so called Rikitake two-disc dynamo system.

2. A geometric formulation of completely integrable systems

The aim of this section is to present the geometry associated to a general completely integrable system. More precisely, using the property that any completely integrable system admits Hamiltonian realizations of Poisson type (see e.g., [12]), we illustrate the associated Poisson geometry, and its relations with the dynamics generated by the system.

In order to do that, let us start by recalling from [12] the Hamiltonian realization procedure of a completely integrable system. For similar Hamilton–Poisson and respectively Nambu–Poisson formulations of completely integrable systems, see e.g., [1], [6], [7], [11], [9].

Recall that a *completely integrable system* is a $\mathcal{C}^\infty$ differential system defined on an open subset $\Omega \subseteq \mathbb{R}^n$,

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