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IFAC PapersOnLine 50-1 (2017) 8115–8120

Stability of discrete fractional linear systems with positive orders ⋆

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Abstract: The problem of the stability of the Caputo–, Riemann-Liouville– and Grünwald-Letnikov–type linear discrete–time systems with fractional positive orders is discussed. We present the method of reducing the fractional order of the considered systems by transforming them to the multi–order linear systems with the partial orders from the interval $(0, 1]$.

Keywords: Stability criteria, discrete–time systems, Z-transform, fractional difference operator.

1. INTRODUCTION

Fractional integrals, derivatives and differences of any order are the basic concepts in the fractional calculus that is a field of applied mathematics. Basic information on fractional calculus, ideas and some applications can be found for example in Podlubny (1999); Kilbas et al. (2006); Kaczorek (2011); Ortigueira and Trujillo (2015); Ostalczyk (2016). One of the most important issue that should be solved for fractional order systems is the analysis of stability. In the case of linear fractional order difference systems the $\mathcal{Z}$-transform can be used as an effective method for the stability analysis, see for instance Mozyrska and Wyrwas (2016); Stanisławski and Latawiec (2013a,b); Abu-Saris and Al-Mdallal (2013); Mozyrska and Wyrwas (2015). There is common for authors to consider fractional systems with positive fractional orders less or equal to one. In the paper we take into account fractional orders that are greater than one. We show that one can reduce the order of the considered systems by transforming them to the systems with the partial orders from the interval $(0, 1]$. Then the results can be applied to transformed systems and consequently, we get the conditions for stability of linear difference systems with fractional orders $\alpha > 0$. The strength of the paper is that we present our result for appropriate systems with the Caputo–, Riemann-Liouville– and Grünwald-Letnikov–type fractional differences with positive orders. We prove that in fact two of them the Riemann-Liouville– and Grünwald-Letnikov–type are equivalent.

The paper is organized as follows. In Section 2 we gather some results needed in the sequel. Section 3 contains the stability analysis of linear difference systems with positive fractional orders. Additionally, similarly as in Busłowicz and Ruszewski (2013); Stanisławski and Latawiec (2013a,b) we prove the conditions connected with eigenvalues of the matrices that define the considered linear difference systems. Finally we present the example in order to illustrate the reworked conditions of stability.

2. OPERATORS AND THEIR $\mathcal{Z}$ – TRANSFORMS

The important role in definitions of fractional operators plays the following sequence of coefficients defined by its values:

$$a^{(\alpha)}(k) := \begin{cases} 1 & \text{for } k = 0 \\ (-1)^k \dfrac{\alpha(\alpha-1)...(\alpha-k+1)}{k!} & \text{for } k \in \mathbb{N}, \end{cases} \tag{1}$$

where $\alpha \in \mathbb{R}$. Since $\frac{\alpha(\alpha-1)...(\alpha-k+1)}{k!} = \binom{\alpha}{k}$, the sequence $\left(a^{(\alpha)}(k)\right)_{k\in\mathbb{N}\cup\{0\}}$ can be rewritten using the generalized binomial $\binom{\alpha}{k}$ as follows $a^{(\alpha)}(k) = (-1)^k\binom{\alpha}{k}$. Let us note that $a^{(\alpha)}$ can be also define in the recurrence way as the following sequence:

$$\begin{aligned} a^{(\alpha)}(0) &:= 1\,, \\ a^{(\alpha)}(k) &:= \left(1 - \frac{\alpha+1}{k}\right) a^{(\alpha)}(k-1)\,, \quad k \in \mathbb{N}\,. \end{aligned} \tag{2}$$

Let us recall that one–sided $\mathcal{Z}$-transform of a sequence $(y(n))_{n\in\mathbb{N}\cup\{0\}}$ is a complex function given by

$$Y(z) := \mathcal{Z}[y](z) = \sum_{k=0}^{\infty} \frac{y(k)}{z^k}\,, \tag{3}$$

where $z \in \mathbb{C}$ denotes a complex number for which the series (3) converges absolutely. It is useful tool for solving difference equations with initial conditions. We treat all discrete functions that they are zero for negative arguments. Note that since $a^{(\alpha)}(k) = (-1)^k\binom{\alpha}{k} = \binom{k-\alpha-1}{k}$, then for $|z| > 1$ and $\alpha \in \mathbb{R}$ we have

$$\mathcal{Z}\left[a^{(\alpha)}\right](z) = \sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}z^{-k} = \left(1 - z^{-1}\right)^{\alpha}\,. \tag{4}$$

⋆ The work was supported by Bialystok University of Technology grant No S/WI/1/2016.

10.1016/j.ifacol.2017.08.1250

2.1 Fractional sums

Let $h > 0$. For a real function $x = x(t)$ the forward h-difference operator is defined as (see Ferreira and Torres (2011))

$$(\Delta_h x)(t) := \frac{x(t+h) - x(t)}{h}$$

and the backward h-difference operator is defined as

$$(\nabla_h x)(t) := \frac{x(t) - x(t-h)}{h}.$$

Then $(\Delta_h x)(t) = (\nabla_h x)(t-h)$. Let $q \in \mathbb{N}$ and $\Delta_h^q := \Delta_h \circ \cdots \circ \Delta_h$ is q-fold application of operator Δ_h, i.e. $\Delta_h^q x = \underbrace{\Delta_h(\Delta_h(\ldots \Delta_h}_{q-\text{times}} x))$ and we write $(\Delta_h^0 x)(t) := x(t)$. Let us notice that $(\Delta_h^q x)(t) = h^{-q} \sum_{k=0}^{q} (-1)^{q-k} \binom{q}{k} x(t+kh)$.

Definition 1. For a real function $x = x(t)$ the *fractional h-sum of order* $\alpha > 0$ is given by

$$\left(\Delta_h^{-\alpha} x\right)(t) := h^{\alpha} \sum_{i=0}^{k} a^{(-\alpha)}(k-i) x(ih) = h^{\alpha} \left(a^{(-\alpha)} * \overline{x}\right)(k),$$

where $t = kh$, $k \in \mathbb{N} \cup \{0\}$, $\overline{x}(k) = x(kh)$ and "$*$" denotes one-sided convolution operator. Additionally, $\left(\Delta_h^0 x\right)(t) := x(t)$.

For simplicity of notation, if $h = 1$, then index h will be omitted and Δ will be written instead if Δ_1. The same concept of notations is assumed for next operators.

Proposition 1. For $t = kh$ let us define $y(k) := \left(\Delta_h^{-\alpha} x\right)(t)$, where $\alpha, h > 0$. Then

$$\mathcal{Z}[y](z) = h^{\alpha} \left(1 - z^{-1}\right)^{-\alpha} X(z), \tag{5}$$

where $X(z) := \mathcal{Z}[\overline{x}](z)$.

Proof. Let $y(k) = \left(\Delta_h^{-\alpha} x\right)(t)$. Then $\mathcal{Z}[y](z) = h^{\alpha} \mathcal{Z}\left[a_k^{(-\alpha)}\right](z) X(z)$. By (4) we see equality (5).

- For $h = 1$ equation (5) is shortly written as
$$\mathcal{Z}\left[\Delta^{-\alpha} x\right](z) = \left(1 - z^{-1}\right)^{-\alpha} X(z).$$

2.2 Caputo–type operator with positive orders

The definition of the Caputo–type fractional h-difference operator can be found, for example, in Mozyrska and Girejko (2013) (for any $h > 0$).

Definition 2. Let $\alpha \in (q-1, q]$, $q \in \mathbb{N}$. The *Caputo–type fractional h-difference operator* Δ_h^{α} of order α of a real function $x = x(t)$ is defined by

$$\left(\Delta_{h,*}^{\alpha} x\right)(t) = \left(\Delta_h^{-(q-\alpha)} \left(\Delta_h^q x\right)\right)(t), \tag{6}$$

where $t = kh$, $k \in \mathbb{N} \cup \{0\}$.

Observe that for $\alpha = q \in \mathbb{N}$ we have $\left(\Delta_{h,*}^{q} x\right)(t) = \left(\Delta_h^q x\right)(t)$.

Proposition 2. For $\alpha \in (q-1, q]$, $q \in \mathbb{N}$ let us define $y(k) := \left(\Delta_{h,*}^{\alpha} x\right)(t)$, where $t = kh$. Then

$$\mathcal{Z}[y](z) = h^{-\alpha} z^q \left(1 - z^{-1}\right)^{\alpha} (X(z) - g(z)), \tag{7}$$

where

$$g(z) = \frac{z}{z-1} \sum_{k=0}^{q-1} (z-1)^{-k} \left(\Delta^k x\right)(0)$$

and $X(z) = \mathcal{Z}[\overline{x}](z)$, $\overline{x}(k) := x(kh)$.

Proof. We state the proof with $h = 1$, as for $h > 0$ it is only simple generalization that is connected with multiplication by h^{α}. By Proposition 1 we have

$$\mathcal{Z}\left[\Delta^{-(q-\alpha)} \left(\Delta^q x\right)\right](z) = \left(1 - z^{-1}\right)^{\alpha - q} \mathcal{Z}\left[\Delta^q x\right](z).$$

Moreover,

$$\mathcal{Z}\left[\Delta^q x\right](z) = (z-1)^q X(z) - z \sum_{k=0}^{q-1} (z-1)^{q-1-k} \left(\Delta^k x\right)(0),$$

where $X(z) = \mathcal{Z}[\overline{x}](z)$. Then

$$\mathcal{Z}\left[\Delta^{-(q-\alpha)} \left(\Delta^q x\right)\right](z) = z^q \left(1 - z^{-1}\right)^{\alpha} (X(z) - g(z)).$$

For orders $\alpha \in (0, 1]$ we have:

$$\mathcal{Z}[y](z) = h^{-\alpha} z \left(1 - z^{-1}\right)^{\alpha} \left(X(z) - \frac{z}{z-1} x(0)\right), \tag{8}$$

where $X(z) = \mathcal{Z}[\overline{x}](z)$ and $\overline{x}(k) := x(kh)$.

2.3 Riemann–Liouville–type operator of positive order

The definition of the fractional h-difference Riemann-Liouville–type operator can be found, for example, in Atıcı and Eloe (2007) (for $h = 1$) or in Bastos et al. (2011) (for any $h > 0$).

Definition 3. Let $\alpha \in (q-1, q]$, $q \in \mathbb{N}$. The *Riemann-Liouville–type fractional h-difference operator* Δ_h^{α} of order α of a real function $x = x(t)$ is defined by

$$\left(\Delta_h^{\alpha} x\right)(t) = \left(\Delta_h^q \left(\Delta_h^{-(q-\alpha)} x\right)\right)(t), \tag{9}$$

where $t = kh$, $k \in \mathbb{N} \cup \{0\}$.

Observe that the Riemann-Liouville–type fractional h-difference operator of order $\alpha = q \in \mathbb{N}$ equals to q-fold application of the forward h-difference operator.

Proposition 3. For $\alpha \in (q-1, q]$, $q \in \mathbb{N}$ let us define $y(k) := \left(\Delta_h^{\alpha} x\right)(t)$, where $t = kh$, Then

$$\mathcal{Z}[y](z) = h^{-\alpha} \left(z^q \left(1 - z^{-1}\right)^{\alpha} X(z) - z \sum_{k=0}^{q-1} (z-1)^{q-k-1} \left(\Delta^k \left(\Delta^{-(q-\alpha)} \overline{x}\right)\right)(0)\right), \tag{10}$$

where $X(z) = \mathcal{Z}[\overline{x}](z)$ and $\overline{x}(k) := x(kh)$.

Proof. We state the proof with $h = 1$. Let $f(k) = \left(\Delta^{-(q-\alpha)} \overline{x}\right)(k)$. Then

$$\mathcal{Z}[y](z) = \mathcal{Z}\left[\Delta^q f\right][z] = (z-1)^q F(z) - z \sum_{k=0}^{q-1} (z-1)^{q-1-k} \left(\Delta^k f\right)(0),$$

where $F(z) = \mathcal{Z}[f](z) = \left(1 - z^{-1}\right)^{\alpha - q} X(z)$. Hence equality (10) holds.

For orders $\alpha \in (0, 1]$ we have:

$$\mathcal{Z}[y](z) = h^{-\alpha} z \left(\left(1 - z^{-1}\right)^{\alpha} X(z) - x(0)\right), \tag{11}$$

where $X(z) = \mathcal{Z}[\overline{x}](z)$ and $\overline{x}(k) := x(kh)$.