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Letnikov-type linear discrete-time systems with fractional positive orders is discussed. We Letthrov-type inteat discrete-time systems with fractional positive orders is discussed. We
present the method of reducing the fractional order of the considered systems by transforming μ present the method of reducing the fractional order of the considered systems by transforming them to the multi-order linear systems with the partial orders from the interval $(0,1]$. Abstract: The problem of the stability of the Caputo–, Riemann-Liouville– and Grünwaldthem to the multi–order linear systems with the partial orders from the interval $(0,1]$. present the method of reducing the fractional order of the considered systems by transforming

© 2017, IFAC (International Federation of Automatic Control) Hosting by Elsevier Ltd. All rights reserved. \mathcal{L} (interaction of the system systems, \mathcal{L} transform, \mathcal{L} transform, \mathcal{L} and $\mathcal{L$ \approx 2011, if the intermediated calculated or theories \sim 1000meth σ (1000meth σ). The interval (1, 1).

Keywords: Stability criteria, discrete–time systems, Z-transform, fractional difference operator. Keywords: Stability criteria, discrete–time systems, Z-transform, fractional difference operator. Keywords: Stability criteria, discrete–time systems, Z-transform, fractional difference operator.

1. INTRODUCTION 1. INTRODUCTION 1. INTRODUCTION

Fractional integrals, derivatives and differences of any order are the basic concepts in the fractional calculus that is a field of applied mathematics. Basic information on fractional calculus, ideas and some applications can on fractional calculus, ideas and some applications can
be found for example in Podlubny (1999); Kilbas et al. (2006) ; Kaczorek (2011) ; Ortigueira and Trujillo (2015) ; Ostalczyk (2016). One of the most important issue that $\frac{1}{2}$ of $\frac{1}{2}$ (2010). One of the most important issue that should be solved for fractional order systems is the analysis of stability. In the case of linear fractional order differ- ϵ or stability. In the case of linear fractional order difference systems the Z -transform can be used as an effective method for the stability analysis, see for instance Mozyrska and Wyrwas (2016); Stanisławski and Latawiec (2013a,b); Abu-Saris and Al-Mdallal (2013); Mozyrska and Wyrwas (2015). There is common for authors to consider fractional Abu-Saris and Al-Mdallal (2013); Mozyrska and Wyrwas and Wyrwas (2016); Stanis�lawski and Latawiec (2013a,b); (2015) . There is common for authors to consider fractional systems with positive fractional orders less or equal to one. In the paper we take into account fractional orders that In the paper we take like account fractional orders that are greater than one. We show that one can reduce the order of the considered systems by transforming them to the systems with the partial orders from the interval $(0, 1]$. Then the results can be applied to transformed systems Then the results can be applied to transformed systems and consequently, we get the conditions for stability of $\lim_{\alpha \to \infty} \frac{d}{d}$ of α is the conditions for stability of the paper is the paper of the paper of $\alpha > 0$. The strength of the paper is that we present our re-The strength of the paper is that we present our re-
sult for appropriate systems with the Caputo–, Riemann-Liouville– and Grünwald-Letnikov–type fractional differ-Enouville– and Grünwald-Letnikov–type fractional differ-
ences with positive orders. We prove that in fact two of them the Riemann-Liouville– and Grünwald-Letnikov– type are equivalent. Fractional integrals, derivatives and differences of any $T_{\rm F}$ and $T_{\rm eff}$ we gather is organized as follows. of them the Riemann-Liouville– and Grunwald-Letnikov–
type are equivalent order are the basic concepts in the fractional calculus that is a field of applied mathematics. Basic information on fractional calculus, ideas and some applications can be found for example in Podlubny (1999); Kilbas et al. (2000); Kaczorek (2011); Ortigueira and Trujino (2015); ζ Ustalczyk (2016). One of the most important issue that $\frac{1}{2}$ should be solved for tractional order systems is the analysis or stability. In the case of linear fractional order differ- ϵ systems the z -transform can be used as an effective method for the stability analysis, see for instance Mozyrska and wyrwas (2010) ; stanisławski and Latawiec $(2013a, 0)$; Abu-Saris and Al-Mdallal (2013); Mozyrska and Wyrwas (2015) . There is common for authors to consider fractional systems with positive fractional orders less or equal to one.
I In the paper we take into account fractional orders that are greater than one. We show that one can requee the order of the considered systems by transforming them to the systems with the partial orders from the interval $(0, 1]$. Then the results can be applied to transformed systems and consequently, we get the conditions for stability of \mathcal{X} ilhear difference systems with fractional orders $\alpha > 0$. The strength of the paper is that we present our resuit for appropriate systems with the Caputo–, Kiemann-Liouville and Grunwald-Letnikov–type fractional differences with positive orders. We prove that in fact two

The paper is organized as follows. In Section 2 we gather The paper is organized as follows. In Section 2 we gather some results needed in the sequel. Section 3 contains the stability analysis of linear difference systems with positive fractional orders. Additionally, similarly as in Busłowicz and Ruszewski (2013); Stanisławski and Lataw-Busłowicz and Ruszewski (2013); Stanisławski and Lataw- $\sum_{i=1}^{\infty}$ is one results needed in the sequel. Section 3 contains the stability analysis of linear difference systems with positive fractional orders. Additionally, similarly as in iec (2013a,b) we prove the conditions connected with eigenvalues of the matrices that define the considered with linear difference systems. Finally we present the example in order to illustrate the reworked conditions of stability. in order to illustrate the reworked conditions of stability. eigenvalues of the matrices that define the considered with with linear difference systems. Finally we present the example

2. OPERATORS AND THEIR Z – TRANSFORMS 2. OPERATORS AND THEIR Z – TRANSFORMS Ω oper atops and their σ transforms

The important role in definitions of fractional operators The important role in definitions of fractional operators
plays the following sequence of coefficients defined by its values: values: values: The important role in definitions of fractional operators plays the following sequence of coefficients defined by its
roluce: p_{values}

$$
a^{(\alpha)}(k) := \begin{cases} 1 & \text{for } k = 0\\ (-1)^k \frac{\alpha(\alpha - 1)...(\alpha - k + 1)}{k!} & \text{for } k \in \mathbb{N}, \end{cases}
$$

1

where $\alpha \in \mathbb{R}$. Since $\frac{\alpha(\alpha-1)...(\alpha-k+1)}{k!} = \binom{\alpha}{k}$, the sequence $(a^{(\alpha)}(k))_{k \in \mathbb{N} \cup \{0\}}$ can be rewritten using the generalized binomial $\binom{\alpha}{k}$ as follows $a^{(\alpha)}(k) = (-1)^k \binom{\alpha}{k}$. Let us note that $a^{(\alpha)}$ can be also define in the recurrence way as the following sequence: $\left(\alpha\right)$ can be also define in the recurrence way as the recurrence way as the recurrence way as the recurrence was the recurrence way as the recurrence was the recurrence was the recurrence was the r where $\alpha \in \mathbb{R}$. Since $\frac{\alpha(\alpha-1)...(\alpha-k+1)}{k!} = \binom{\alpha}{k}$, the sequence where $\alpha \in \mathbb{R}$. Since $\frac{\alpha}{k!} = {k \choose k}$, the sequence $(a^{(\alpha)}(k))_{k \in \mathbb{N} \cup \{0\}}$ can be rewritten using the generalized where $\alpha \in \mathbb{R}$. Since $\frac{\alpha}{k!}$ $\alpha(k)$ $(a^{(x)}(k))_{k \in \mathbb{N} \cup \{0\}}$ can be rewritten using the generalized
historial (a) or follows $a^{(0)}(k)$ (b) (a) is (a) . that $a^{(\alpha)}$ can be also define in the recurrence way as the following sequence: where $\alpha \in \mathbb{R}$. Since $\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} = \binom{\alpha}{k}$, the sequence $(a^{(\alpha)}(k))_{k \in \mathbb{N} \cup \{0\}}$ can be rewritten using the generalized

$$
a^{(\alpha)}(0) := 1,
$$

\n
$$
a^{(\alpha)}(k) := \left(1 - \frac{\alpha + 1}{k}\right) a^{(\alpha)}(k - 1), \quad k \in \mathbb{N}.
$$
\n(2)

Let us recall that one–sided Z -transform of a sequence Let us recall that one–sided \sum transform of a sequence $(y(n))_{n \in \mathbb{N} \cup \{0\}}$ is a complex function given by $(y(n))_{n\in\mathbb{N}\cup\{0\}}$ is a complex function given by

$$
Y(z) := \mathcal{Z}[y](z) = \sum_{k=0}^{\infty} \frac{y(k)}{z^k},
$$
 (3)
where $z \in \mathbb{C}$ denotes a complex number for which the series

where $z \in \mathbb{C}$ denotes a complex number for which the series where $z \in \mathbb{C}$ denotes a complex number for which the series (3) converges absolutely. It is useful tool for solving difference equations with initial conditions. We treat all discrete functions that they are zero for negative arguments. Note that since $a^{(\alpha)}(k) = (-1)^k {\binom{\alpha}{k}} = {\binom{k-\alpha-1}{k}}$, then for $|z| > 1$ and $\alpha \in \mathbb{R}$ we have $\frac{1}{2}$ that since $a^{(\alpha)}(k) = (-1)^k {(\alpha) \choose k} = {(\alpha - \alpha)^k \choose k}$, then for $|z| > 1$ $k=0$
mber for which the series (3) converges absolutely. It is useful tool for solving difference equations with initial conditions. We treat all discrete functions that they are zero for negative arguments. Note = �^k−α−¹

$$
\mathcal{Z}\left[a^{(\alpha)}\right](z) = \sum_{k=0}^{\infty} (-1)^k {(\alpha) \choose k} z^{-k} = \left(1 - z^{-1}\right)^{\alpha} . \tag{4}
$$

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2.1 Fractional sums

Let $h > 0$. For a real function $x = x(t)$ the forward hdifference operator is defined as (see Ferreira and Torres (2011)

$$
(\Delta_h x)(t) := \frac{x(t+h) - x(t)}{h}
$$

and the backward h-difference operator is defined as

$$
(\nabla_h x)(t) := \frac{x(t) - x(t - h)}{h}.
$$

Then $(\Delta_h x)(t) = (\nabla_h x)(t - h)$. Let $q \in \mathbb{N}$ and $\Delta_h^q := \Delta_h \circ$ $\cdots \circ \Delta_h$ is q-fold application of operator Δ_h , i.e. $\Delta_h^q x =$ $\Delta_h(\Delta_h(\ldots \Delta_h x))$ and we write $(\Delta_h^0 x)(t) := x(t)$. Let us q −times

notice that $(\Delta_h^q x)(t) = h^{-q} \sum_{k=0}^q (-1)^{q-k} {q \choose k} x(t + kh)$. Definition 1. For a real function $x = x(t)$ the fractional h-sum of order $\alpha > 0$ is given by

$$
\left(\Delta_h^{-\alpha}x\right)(t) := h^{\alpha}\sum_{i=0}^k a^{(-\alpha)}(k-i)x(ih) = h^{\alpha}\left(a^{(-\alpha)}*\overline{x}\right)(k),
$$

where $t = kh, k \in \mathbb{N} \cup \{0\}, \overline{x}(k) = x(kh)$ and "*" denotes one-sided convolution operator. Additionally, $(\Delta_h^0 x)(t) :=$ $x(t)$.

For simplicity of notation, if $h = 1$, then index h will be omitted and Δ will be written instead if Δ_1 . The same concept of notations is assumed for next operators.

Proposition 1. For $t = kh$ let us define $y(k) := (\Delta_h^{-\alpha} x)(t)$, where $\alpha, h > 0$. Then

$$
\mathcal{Z}[y](z) = h^{\alpha} (1 - z^{-1})^{-\alpha} X(z),
$$

where $X(z) := \mathcal{Z}[\overline{x}](z).$ (5)

Proof. Let $y(k) = (\Delta_h^{-\alpha} x)(t)$. Then $\mathcal{Z}[y](z) =$ $h^{\alpha} \mathcal{Z} \left[a_k^{(-\alpha)} \right] (z) X(z)$. By (4) we see equality (5).

• For $h = 1$ equation (5) is shortly written as

$$
\mathcal{Z}\left[\Delta^{-\alpha}x\right](z) = \left(1 - z^{-1}\right)^{-\alpha} X(z).
$$

2.2 Caputo–type operator with positive orders

The definition of the Caputo–type fractional h -difference operator can be found, for example, in Mozyrska and Girejko (2013) (for any $h > 0$).

Definition 2. Let $\alpha \in (q-1,q], q \in \mathbb{N}$. The Caputo-type fractional h-difference operator Δ_h^{α} of order α of a real function $x = x(t)$ is defined by

$$
\left(\Delta_{h,*}^{\alpha}x\right)(t) = \left(\Delta_h^{-(q-\alpha)}\left(\Delta_h^q x\right)\right)(t),\tag{6}
$$

where $t = kh, k \in \mathbb{N} \cup \{0\}.$

Observe that for $\alpha = q \in \mathbb{N}$ we have $(\Delta_{h,*}^q x)(t) =$ $\left(\Delta_h^q x\right)(t)$.

Proposition 2. For $\alpha \in (q-1,q], q \in \mathbb{N}$ let us define $y(k) := \left(\Delta_{h,*}^{\alpha} x\right)(t)$, where $t = kh$. Then

$$
\mathcal{Z}[y](z) = h^{-\alpha} z^q \left(1 - z^{-1}\right)^{\alpha} \left(X(z) - g(z)\right) ,\qquad(7)
$$
 here

where

$$
g(z) = \frac{z}{z-1} \sum_{k=0}^{q-1} (z-1)^{-k} (\Delta^k x) (0)
$$

and
$$
X(z) = \mathcal{Z}[\overline{x}](z), \overline{x}(k) := x(kh).
$$

Proof. We state the proof with $h = 1$, as for $h > 0$ it is only simple generalization that is connected with multiplication by h^{α} . By Proposition 1 we have

$$
\mathcal{Z}\left[\Delta^{-(q-\alpha)}\left(\Delta^q x\right)\right](z) = \left(1-z^{-1}\right)^{\alpha-q} \mathcal{Z}\left[\Delta^q x\right](z).
$$

Moreover,

$$
\mathcal{Z}\left[\Delta^q x\right](z) = (z-1)^q X(z) - z \sum_{k=0}^{q-1} (z-1)^{q-1-k} \left(\Delta^k x\right)(0),
$$

where
$$
X(z) = \mathcal{Z}[\overline{x}](z)
$$
. Then
\n
$$
\mathcal{Z}\left[\Delta^{-(q-\alpha)}\left(\Delta^q x\right)\right](z) = z^q \left(1 - z^{-1}\right)^{\alpha} \left(X(z) - g(z)\right).
$$

For orders $\alpha \in (0, 1]$ we have:

$$
\mathcal{Z}[y](z) = h^{-\alpha} z \left(1 - z^{-1}\right)^{\alpha} \left(X(z) - \frac{z}{z - 1} x(0)\right), \quad (8)
$$

where $X(z) = \mathcal{Z}[\overline{x}](z)$ and $\overline{x}(k) := x(kh)$.

2.3 Riemann–Liouville–type operator of positive order

The definition of the fractional h-difference Riemann-Liouville–type operator can be found, for example, in Atıcı and Eloe (2007) (for $h = 1$) or in Bastos et al. (2011) (for any $h > 0$).

Definition 3. Let $\alpha \in (q-1, q], q \in \mathbb{N}$. The Riemann-Liouville–type fractional h-difference operator Δ_h^α of order α of a real function $x = x(t)$ is defined by

$$
\left(\Delta_h^{\alpha}x\right)(t) = \left(\Delta_h^q\left(\Delta_h^{-(q-\alpha)}x\right)\right)(t),\tag{9}
$$

where $t = kh, k \in \mathbb{N} \cup \{0\}.$

Observe that the Riemann-Liouville–type fractional hdifference operator of order $\alpha = q \in \mathbb{N}$ equals to q-fold application of the forward h-difference operator.

Proposition 3. For $\alpha \in (q-1, q], q \in \mathbb{N}$ let us define $y(k) := (\Delta_h^{\alpha} x)(t)$, where $t = kh$, Then

$$
\mathcal{Z}\left[y\right](z) = h^{-\alpha} \left(z^q \left(1 - z^{-1}\right)^{\alpha} X(z) - z \sum_{k=0}^{q-1} (z-1)^{q-k-1} \left(\Delta^k \left(\Delta^{-(q-\alpha)} \overline{x}\right)\right)(0)\right),
$$
\n
$$
(10)
$$

where $X(z) = \mathcal{Z}[\overline{x}](z)$ and $\overline{x}(k) := x(kh)$.

Proof. We state the proof with $h = 1$. Let $f(k) =$ $(\Delta^{-(q-\alpha)}\overline{x})$ (k). Then

$$
\mathcal{Z}[y](z) = \mathcal{Z}[\Delta^q f][z] = (z-1)^q F(z) - z \sum_{k=0}^{q-1} (z-1)^{q-1-k} (\Delta^k f)(0),
$$

where $F(z) = \mathcal{Z}[f](z) = (1-z^{-1})^{\alpha-q} X(z)$. Hence equality (10) holds.

For orders $\alpha \in (0,1]$ we have:

$$
\mathcal{Z}[y](z) = h^{-\alpha} z \left(\left(1 - z^{-1} \right)^{\alpha} X(z) - x(0) \right), \quad (11)
$$

where $X(z) = \mathcal{Z}[\overline{x}](z)$ and $\overline{x}(k) := x(kh)$.

ِ متن کامل مقا<mark>ل</mark>ه

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