Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 50-1 (2017) 3099–3104

Internally Positive Representations and Stability Analysis of Linear Difference Systems with Multiple Delays

Vittorio De Iuliis * **Alfredo Germani** * **Costanzo Manes** *

* *Dipartimento di Ingegneria e Scienze dell'Informazione, e Matematica, Università degli Studi dell'Aquila, Via Vetoio, 67100 Coppito (AQ), Italy (e-mail: vittorio.deiuliis@graduate.univaq.it, {alfredo.germani–costanzo.manes}@univaq.it).*

Abstract: This work introduces the Internally Positive Representation of linear continuous-time difference systems with multiple delays. The technique, originally developed for delay-free linear systems, and recently extended to differential systems with multiple delays, proved to be a useful tool to exploit results available for positive systems also for systems that are not positive. As a result, a sufficient delay-independent stability condition for difference systems with constant delays is given, which is shown to be less conservative than similar existing results.

Keywords: Linear systems, Systems with delays, Positive systems, Delay-independent stability.

1. INTRODUCTION

Linear continuous-time difference systems with multiple delays are systems governed by equations of the form

$$x(t) = \sum_{k=1}^{m} A_k x(t-\delta_k). \tag{1}$$

These systems are well known in literature for their importance in modeling many physical phenomena of wide interest such as networks of conservation laws, resulting in a vast amount of possible applications, as road traffic and electric transmission models (see Bellman and Cooke (1963), Melchor-Aguilar (2016), Damak et al. (2016a), and references therein). Moreover, continuous-time difference equations arise in coupled differential-difference equations and neutral systems, and hence their stability properties have been deeply investigated and their impact on those classes of systems has been extensively discussed, e.g., in Pepe (2003), Pepe and Verriest (2003).

This paper presents two main results concerning linear continuous-time difference systems with multiple delays. The first one is the extension of the Internally Positive Representation of systems (IPR, in short) to this class of delay systems. The IPR technique was introduced in the linear delay-free framework for discrete-time systems in Germani et al. (2007, 2010), Cacace et al. (2012b) and continuous-time systems in Cacace et al. (2012a, 2014), to extend properties only available for positive systems also to systems which are not positive (*arbitrary* systems, hereafter). This has proved to be very useful in the time-delay setting, where the IPR approach has recently been applied to differential systems with multiple delays (see Conte et al. (2017)) resulting in a delay-independent sufficient condition for the stability of arbitrary systems of that class. Hence, the results obtained in the differential case suggested to investigate the possibility of extending the IPR construction also to other classes of delay systems, as continuous-time difference systems with multiple delays. Here we come to the second main result of the paper: a sufficient condition for the delay-independent exponential stability of such systems. The proposed condition, obtained through the IPR technique, is easy to check and is shown to be less conservative than a well-known sufficient (norm-based) condition of comparable simplicity.

The remainder of the paper is structured as follows. Section 2 starts with some preliminary definitions on notations and mathematical tools arising in positive systems, followed by the introduction of the Internally Positive Representation for the class of systems under investigation. Section 3 gives some background on stability concepts for difference systems with delays and then focuses on the stability condition obtained via the IPR approach, which is compared to similar available results. Section 4 illustrates the new results by means of numerical examples, and Section 5 presents the conclusions and some ideas for future works.

2. INTERNALLY POSITIVE REPRESENTATION OF DIFFERENCE SYSTEMS WITH DELAYS

2.1 Notations and positive representations of matrices

Notations. $\mathbb{R}_+$ is the set of nonnegative real numbers. $\mathbb{R}^n_+$ is the nonnegative orthant of $\mathbb{R}^n$. $\mathbb{R}^{m\times n}_+$ is the cone of positive $m \times n$ matrices. I_n is the $n \times n$ identity matrix. The Banach space of all piecewise right-continuous and bounded functions defined on $[a,b]$ with values in $\mathbb{R}^n$ is denoted by $\mathcal{PC}([a,b],\mathbb{R}^n)$, and is endowed with the uniform convergence norm $\|\cdot\|_\infty$. $\sigma(M)$ and $\rho(M)$ denote the spectrum and the spectral radius of M, respectively. M is said to be *Schur-stable* if $\sigma(M) \subset \{z \in \mathbb{C} : |z| < 1\}$ or, equivalently, if $\rho(M) < 1$. Finally, $\underline{m} = \{1, 2, \dots, m\}$.

10.1016/j.ifacol.2017.08.684

Positive representations of matrices. Given a matrix (or vector) $M \in \mathbb{R}^{m\times n}$, the symbols M^+, M^- stand for the componentwise *positive* and *negative* parts of M, while $|M|$ denotes its componentwise absolute value. Then $M = M^+ - M^-$ and $|M| = M^+ + M^-$.
Let $\Delta_n = [I_n \ -I_n] \in \mathbb{R}^{n\times 2n}$. The following definitions are taken from Germani et al. (2010), Cacace et al. (2012b).

Definition 1. A positive representation of a vector $x \in \mathbb{R}^n$ is any vector $\tilde{x} \in \mathbb{R}^{2n}_+$ such that

$$x = \Delta_n \tilde{x}. \tag{2}$$

The min-positive representation of a vector $x \in \mathbb{R}^n$ is the positive vector $\pi(x) \in \mathbb{R}^{2n}_+$ defined as

$$\pi(x) = \begin{bmatrix} x^+ \\ x^- \end{bmatrix}. \tag{3}$$

The min-positive representation of a matrix $M \in \mathbb{R}^{m\times n}$ is the positive matrix $\Pi(M) \in \mathbb{R}^{2m\times 2n}_+$ defined as

$$\Pi(M) = \begin{bmatrix} M^+ & M^- \\ M^- & M^+ \end{bmatrix}. \tag{4}$$

Proposition 2. For any $x \in \mathbb{R}^n$ and $M \in \mathbb{R}^{m\times n}$, the following properties hold true:

(a) $x = \Delta_n \pi(x)$;
(b) $\Delta_m \Pi(M) = M\Delta_n$, so that $\Delta_m \Pi(M)\pi(x) = Mx$.

2.2 Internally Positive Difference Systems with Delays

In this work, we deal with linear continuous-time systems governed by difference equations with multiple delays. We will denote such a system by $S = \{\{A_k\}_{k=1}^m, B, C, D\}_{n,p,q}$, with S having the following form:

$$\begin{aligned} x(t) &= \sum_{k=1}^{m} A_k x(t-\delta_k) + Bu(t), && t \geq t_0, \\ y(t) &= Cx(t) + Du(t), && \\ x(t) &= \phi(t - t_0), && t \in [t_0 - \delta, t_0), \end{aligned} \tag{5}$$

where $x(t) \in \mathbb{R}^n$ is the system variable, $u(t) \in \mathbb{R}^p$ is the input, with $u \in \mathcal{PC}([t_0, +\infty), \mathbb{R}^p)$, $y(t) \in \mathbb{R}^q$ is the output, and $\phi \in \mathcal{PC}([-\delta, 0), \mathbb{R}^n)$ is the initial state function. Moreover, δ_k is a constant delay for any $k \in \underline{m}$, with $0 < \delta_1 < \cdots < \delta_k < \cdots < \delta_m = \delta$. It follows that: $A_k \in \mathbb{R}^{n\times n}$, for any $k \in \underline{m}$, $B \in \mathbb{R}^{n\times p}, C \in \mathbb{R}^{q\times n}, D \in \mathbb{R}^{q\times p}$.
The trajectories of S will be denoted by:

$$(x(t), y(t)) = \Phi_S(t, t_0, \phi, u). \tag{6}$$

We investigate the possibility of representing a system that is not positive (an *arbitrary* system) by means of an internally positive system of larger dimensions. A strong motivation behind this idea is the fact that some appealing stability results have been established for positive delay systems which, in general, don't apply to their arbitrary counterpart. A system is internally positive if its trajectories can only assume nonnegative values provided that the initial conditions and the inputs are nonnegative. The following definition rigorously catches this property.

Definition 3. $S = \{\{A_k\}_{k=1}^m, B, C, D\}_{n,p,q}$ is defined to be internally positive if the following implication holds:

$$\left\{ \begin{array}{l} \phi \in \mathcal{PC}([-\delta, 0), \mathbb{R}^n_+) \\ u \in \mathcal{PC}([t_0, +\infty), \mathbb{R}^p_+) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} x(t) \in \mathbb{R}^n_+, \\ y(t) \in \mathbb{R}^q_+, \end{array} \ \forall t \geq t_0 \right\}. \tag{7}$$

A straightforward extension of the results given in Di Loreto and Loiseau (2012) leads to the next Lemma, which provides necessary and sufficient conditions for a system S to be positive.

Lemma 4. $S = \{\{A_k\}_{k=1}^m, B, C, D\}_{n,p,q}$ is internally positive if and only if A_k is nonnegative for any $k \in \underline{m}$, and B, C, D are nonnegative.

2.3 Internally Positive Representations

The Internally Positive Representation (IPR) of an arbitrary system has been introduced in the delay-free discrete-time framework in Germani et al. (2007, 2010), Cacace et al. (2012b), and then extended to delay-free continuous-time systems in Cacace et al. (2012a, 2014). More recently, the technique has been applied to differential systems with multiple delays (see Conte et al. (2017)). Here we extend the IPR construction to continuous-time difference systems with multiple delays.

Definition 5. An *Internally Positive Representation* (IPR) of a delay system $S = \left\{\{A_k\}_{k=1}^m, B, C, D\right\}_{n,p,q}$ is an internally positive system $\widetilde{S} = \left\{\{\widetilde{A}_k\}_{k=1}^m, \widetilde{B}, \widetilde{C}, \widetilde{D}\right\}_{\tilde{n},\tilde{p},\tilde{q}}$ together with four transformations $\{T_X^f, T_X^b, T_U, T_Y\}$,

$$\begin{array}{ll} T_X^f : \mathbb{R}^n \mapsto \mathbb{R}^{\tilde{n}}_+, & T_X^b : \mathbb{R}^{\tilde{n}}_+ \mapsto \mathbb{R}^n, \\ T_U : \mathbb{R}^p \mapsto \mathbb{R}^{\tilde{p}}_+, & T_Y : \mathbb{R}^{\tilde{q}}_+ \mapsto \mathbb{R}^q, \end{array} \tag{8}$$

such that $\forall(\phi, u) \in \mathcal{PC}([-\delta, 0), \mathbb{R}^n) \times \mathcal{PC}([t_0, +\infty), \mathbb{R}^p)$, $\forall t_0 \in \mathbb{R}$, the following implication holds:

$$\begin{array}{l} \left\{ \begin{array}{ll} \tilde{\phi}(\tau) = T_X^f\big(\phi(\tau)\big), & \forall \tau \in [-\delta, 0) \\ \tilde{u}(t) = T_U\big(u(t)\big), & \forall t \geq t_0 \end{array} \right\} \\ \qquad \Longrightarrow \left\{ \begin{array}{l} x(t) = T_X^b\big(\tilde{x}(t)\big), \\ y(t) = T_Y\big(\tilde{y}(t)\big), \end{array} \ \forall t \geq t_0 \right\}. \end{array} \tag{9}$$

where

$$(x(t), y(t)) = \Phi_S(t, t_0, \phi, u)$$
$$(\tilde{x}(t), \tilde{y}(t)) = \Phi_{\tilde{S}}(t, t_0, \tilde{\phi}, \tilde{u}).$$

The behaviour of the Internally Positive Representation technique is depicted in Figure 1.

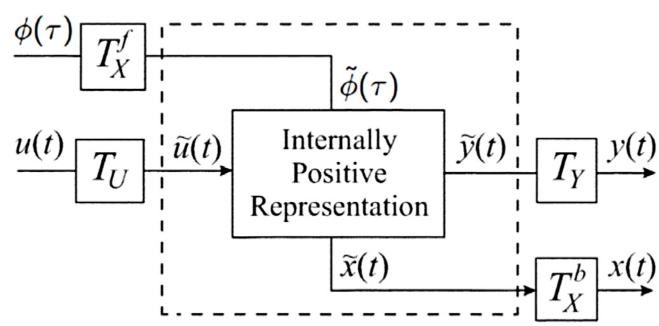

Fig. 1. Block diagram of the IPR in Definition 5.

The maps T_U and T_Y in (8) are the input and output transformations of the IPR, respectively, while T_X^f and T_X^b are the *forward* and *backward* state transformations, respectively; for consistency, the *backward* map T_X^b must be a left-inverse of the *forward* map T_X^f, i.e. $x = T_X^b\big(T_X^f(x)\big)$, $\forall x \in \mathbb{R}^n$. The implication (9) can be further clarified as