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Wirtinger-based Exponential Stability for Time-Delay Systems [1]

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Abstract: This paper deals with the exponential stabilization of a time-delay system with an average of the state as the output. A general stability theorem with a guaranteed exponential decay-rate based on a Wirtinger-based inequality is provided. Variations of this theorem for synthesis of a controller or for an observer-based control is derived. Some numerical comparisons are proposed with existing theorems of the literature and comparable results are obtained but with an extension to stabilization.

Keywords: Time-delay systems, Exponential stability, Lyapunov methods, Wirtinger inequality, Controller and observer synthesis

1. INTRODUCTION

Time-delay systems may arise in practice for many reasons. For example, it appears in mechanical modeling like vibration absorber (see Olgac and Holm-Hansen (1994)) or delayed resonator (see Gu et al. (2003)) which are intrinsically with delay and neglecting it leads to an over-simplification of the initial problem. That is why it is important to have a theory which can provide a framework to work with. Indeed, although time-delay systems are a class of dynamical systems widely studied in control theory, the honored method like root-locus to assess stability are not straightforward, particularly to provide robust stability criteria.

Three main approaches have been developed to study the stability of the such equations. The first one relies on the characteristic equation (see Sipahi et al. (2011) and references therein and Breda (2006)) and pole location. These techniques give nearly the exact stability conditions but suffer from several drawbacks. First of all, as they are based on pole location approximations, they are not appropriated for uncertain and/or time-varying delay systems. Furthermore, these approaches could not also be used easily for the design of controllers or observers.

Other approaches have been developed based either on the robust approach or Lyapunov techniques. The robust approach consists of merging the delay uncertainty into an uncertain set and use classical robust analysis as Small Gain Theorem (Fridman et al. (2008)), Quadratic Separation (Gouaisbaut and Peaucelle (2006)), Integral Quadratic Constraints (Kao and Rantzer (2007)). Techniques based on Lyapunov-Krasovskii functionals uses the LMI framework developed in the book by Boyd et al. (1994). This method enables exponential convergence with a guaranteed decay rate, robust analysis, synthesis of controllers and extension to multiple time-varying delay systems.

Despite these advantages, this approach is very conservative. The complete Lyapunov-Krasovskii functional is known (Kharitonov and Zhabko (2003)) but too complex to be efficiently solved and even studied. A first step is to introduce a simplified functional. Some works have been done (for example by Seuret and Gouaisbaut (2015)) on how to relax the problem such that the conservatism introduced by the choice of the Lyapunov-Krasovskii functional is measured. The second step is to use integral inequalities to transform some non-manageable terms like $\int_{t-h}^{t} e^{-2\alpha s} x^\top(t+s)Rx(t+s)ds$ into an expression suitable to be transformed into LMIs. This last step is important because there exists powerful and efficient algorithm to find solutions of LMIs in polynomial time. The commonly used inequalities in the two last steps are described by Gu et al. (2003) and rely for most of them on Jensen's inequality. An important amount of papers have been dedicated to reduce the conservatism induced by such inequalities. Recently, Seuret and Gouaisbaut (2013) introduced a Wirtinger-based inequality, known to be less conservative. The present paper uses this framework to state the exponential convergence with a guaranteed decay rate and synthesis of controllers.

Two approaches have been widely used in the literature to assess the exponential stability. The first one relies on a change of variable $z(t) = e^{\alpha t}x(t)$ and it can be proven that establishing asymptotic stability of z implies an exponential stability of x with a decay rate of α (Seuret et al. (2004)). The second one is based on some modified Lyapunov-Krasovskii functionals which incorporate in their structures the exponential rate.

Since one of the first article by Mori et al. (1982) on exponential convergence of time-delay systems, several exponential estimates emerged from the literature: Mondie and Kharitonov (2005), Xu et al. (2006) or more recently Trinh et al. (2016). But only a few of them used the Wirtinger-based inequality developed by Seuret and Gouaisbaut (2013) to help synthesize observers or controllers for a discrete or distributed delay system. The aim of this article

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is to stabilize a specific class of time-delay systems as described in the problem statement using this inequality.

In Section 2, the problem is stated and some useful lemmas are reminded. Then in Section 3, an extension of exponential stability theorems with a Wirtinger-based inequality is introduced. The general results of the previous section are used for the computation of a feedback gain for a given system in Section 4 while Section 5 is dedicated to the design of an observer-based control. Finally, in the last section, a numerical comparison of efficiency between classical theorems and the one derived in this paper is performed.

Notations. Throughout the paper, $\mathbb{R}^n$ stands for the n dimensional Euclidian space, $\mathbb{R}^{n\times m}$ for the set of all $n\times m$ matrices. $\mathbb{S}^n$ is the subset of $\mathbb{R}^{n\times n}$ of symmetric matrices such that $P \in \mathbb{S}^n_+$ or equivalently $P \succ 0$ denotes a symmetric positive definite matrix. For any square matrices A and B, the operations 'He' and 'diag' are defined as follow: $\mathrm{He}(A) = A + A^\top$ and $\mathrm{diag}(A, B) = \left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$. The notations I_n and $0_{n\times m}$ denote the n by n identity matrix and the null matrix of size $n \times m$. The state variable x can be represented using the Shimanov notation (Kolmanovskii and Myshkis (2013)): $x_t : \begin{cases} [-h, 0] \to \mathbb{R}^n \\ \tau \mapsto x(t+\tau) \end{cases}$

2. PROBLEM STATEMENT

2.1 System data

The system to be controlled is the following one:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & \forall t \geqslant 0, \\ y(t) = C\dfrac{1}{h}\displaystyle\int_{-h}^{0} x_t(s)ds, & \forall t \geqslant 0, \\ x(t) = \phi(t), & \forall t \in [-h, 0], \end{cases} \tag{1}$$

with $x(t) \in \mathbb{R}^n$ the instantaneous state vector, h the time delay, ϕ the initial state function and A, B, C three matrices of appropriate dimensions. Then, the output is not the instantaneous state but its average on a sliding window of time $[t-h, t]$, which differs significantly from classical control problems. Numerous measurement tools, in electronics for example, are measuring an average and not the instantaneous state.

The purpose of this paper is to find a control input u computed only with the output measurement vector y such that System (1) is exponentially stable with a decay rate of at least $\alpha \geqslant 0$. First of all, we recall the definition of exponential stability extended to time-delay systems:

Definition 1. (Chen and Zheng (2007)). System (1) is said to be α-stable if there exists $\alpha \geqslant 0$ and $\gamma \geqslant 1$ such that for every solution x of (1) with a differentiable initial condition ϕ defined on $[-h; 0]$, the following exponential estimate holds:

$$\forall t > 0, |x(t)| \leqslant \gamma e^{-\alpha t} \|\phi\|_W \tag{2}$$

where

$$\|\phi\|_W = \max\{||\phi||_h, ||\dot{\phi}||_h\} \text{ and } \|\phi\|_h = \sup_{\theta\in[-h,0]} \|\phi(\theta)\|$$

Remark 1. The norm $\|\cdot\|_W$ is sightly different from the one of Mondie and Kharitonov (2005) who do not consider a norm depending on the derivative $\dot{x}$. This problem has also been dealt by Fridman (2014) by introducing the sum and not the maximum. These definitions are nevertheless equivalent.

2.2 Preliminary Results

We recall two lemmas useful in the sequel. The first lemma, introduced by Seuret and Gouaisbaut (2013) proposes an integral inequality which is used in the proof of the main theorem.

Lemma 1. (Wirtinger-based inequality). For a given matrix $R \in \mathbb{S}^n_+$, the following inequality holds for all continuously differentiable function x in $[t-h, t] \to \mathbb{R}^n$:

$$\int_{t-h}^{t} \dot{x}^\top(s)R\dot{x}(s)ds \geqslant \frac{1}{h}\xi^\top(t)F_2^\top \tilde{R}F_2\xi(t),$$

where

$$F_2 = \begin{bmatrix} I_n & -I_n & 0_n \\ I_n & I_n & -2I_n \end{bmatrix}, \qquad \tilde{R} = \mathrm{diag}(R, 3R),$$

$$\xi(t) = \left[x^\top(t) \; x^\top(t-h) \; \frac{1}{h}\int_{-h}^{0} x^\top(s)ds\right]^\top.$$

The second lemma, called Finsler's lemma, is widely used to cope with non linearities in LMIs.

Lemma 2. (Ebihara et al. (2015))
For any $Q \in \mathbb{S}^n$ and $M \in \mathbb{R}^{p\times n}$, the three following properties are equivalent:

(1) $x^\top Qx \prec 0$ for all $x \in \mathbb{R}^n$ such that $Mx = 0$,
(2) $\exists Y \in \mathbb{R}^{n\times p}, Q + \mathrm{He}\left(M^\top Y\right) \prec 0$,
(3) ${M^\perp}^\top QM^\perp \prec 0$ where $MM^\perp = 0$.

3. EXPONENTIAL STABILITY

Considering a feedback on System (1), i.e. $u(t) = Ky(t)$, it is possible to transform our system into a more general one:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x_t(-h) + A_D \displaystyle\int_{-h}^{0} x_t(s)ds, \; \forall t \geqslant 0, \\ x(t) = \phi(t), \qquad \forall t \in [-h, 0], \end{cases} \tag{3}$$

with $x(t) \in \mathbb{R}^n$ the instantaneous state vector and matrices A, A_d and A_D of appropriate dimensions.

Based on the lemmas recalled above, we propose a first exponential stability result for the previous system.

Theorem 1. Assume that, for given $h > 0$ and $\alpha \geqslant 0$, there exist matrices $P \in \mathbb{S}^{2n}$, $R, S \in \mathbb{S}^n_+$ and $Y \in \mathbb{R}^{n\times 4n}$ and a positive real β_1 such that the following LMIs are satisfied:

$$P_2 = P + \tfrac{e^{-2\alpha h}}{h}\mathrm{diag}(0_n, S) + \tfrac{4\alpha^2 h}{e^{2\alpha h}-2h\alpha-1}\left[\begin{smallmatrix} h^2R & -hR \\ -hR & R \end{smallmatrix}\right] - \beta_1 \mathrm{diag}(I_n, 0_n) \succ 0, \tag{4}$$

$$\Phi(\alpha, h) + \mathrm{He}\left(F_4^\top Y\right) \prec 0, \tag{5}$$

with

$$\Phi(\alpha, h) = \mathrm{He}\left(F_1^\top P(F_0 + \alpha F_1)\right) + \bar{S} + h^2F_3^\top RF_3 - e^{-2\alpha h}F_2^\top \tilde{R}F_2,$$