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Physica A 387 (2008) 1225–1246

PHYSICA A

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# Application of the Beck model to stock markets: Value-at-Risk and portfolio risk assessment

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Received 22 February 2007; received in revised form 17 August 2007
Available online 13 October 2007

**Abstract**

We apply the Beck model, developed for turbulent systems that exhibit scaling properties, to stock markets. Our study reveals that the Beck model elucidates the properties of stock market returns and is applicable to practical use such as the Value-at-Risk estimation and the portfolio analysis. We perform empirical analysis with daily/intraday data of the S&P500 index return and find that the volatility fluctuation of real markets is well-consistent with the assumptions of the Beck model: The volatility fluctuates at a much larger time scale than the return itself and the inverse of variance, or "inverse temperature", $\beta$ obeys $\Gamma$-distribution. As predicted by the Beck model, the distribution of returns is well-fitted by $q$-Gaussian distribution of Tsallis statistics. The evaluation method of Value-at-Risk (VaR), one of the most significant indicators in risk management, is studied for $q$-Gaussian distribution. Our proposed method enables the VaR evaluation in consideration of tail risk, which is underestimated by the variance–covariance method. A framework of portfolio risk assessment under the existence of tail risk is considered. We propose a multi-asset model with a single volatility fluctuation shared by all assets, named the single $\beta$ model, and empirically examine the agreement between the model and an imaginary portfolio with Dow Jones indices. It turns out that the single $\beta$ model gives good approximation to portfolios composed of the assets with non-Gaussian and correlated returns.


*PACS:* 89.65.Gh; 02.50.Ey; 05.10.Gg

*Keywords:* Econophysics; Stock markets; Value-at-Risk; Portfolio risk assessment; Tsallis statistics

## 1. Introduction

Nowadays, the improvement of risk management systems becomes a crucial issue for financial institutions. The fundamental feature of risk management is an estimation of risk and control of its amount within a limited risk buffer. The assets exposed to risk, such as stocks, bonds, and loans are called risk assets. Risk assets sometimes fluctuate unpredictably but such fluctuations are mutually not independent, but correlated. The amount of risk corresponding to risk assets is measured by statistical methods such as Value-at-Risk explained below. Based on the estimated risk, financial institutions have to reserve a capital with which the loss would be compensated when the risk is realized. As a capital, or a risk buffer, is limited, control of the total amount of risk and the allocation strategy of the risk buffer

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0378-4371/$ - see front matter 
doi:10.1016/j.physa.2007.10.023

between risk assets to maximize earnings become important problems. In order to handle such problems the portfolio theory offers an established framework.

Value-at-Risk (VaR) is one of the risk estimation methods widely employed in the practice of risk management [1, 2]. It was propagated by the BIS regulation, where it was adopted as the standard method for evaluating market risk [3]. The brief definition of VaR is described as follows: a VaR with confidence level $c$ for the period of $t$ days means that the losses larger than the amount of the VaR occurs with probability $100(1-c)\%$ during $t$ days. The current standards of VaR measurement are the variance–covariance method and the Monte Carlo method. Variance–covariance method (VC) evaluates a VaR with the assumption of Gaussian distributed returns. Its name comes from the fact that a variance–covariance matrix of assets plays an important role in this method. Based on the portfolio theory, it allows us intuitive analysis and easy calculation of risk, which has enabled VC to propagate early in practice. The VC has a problem, however, that it ignores fluctuations of volatility (heteroskedasticity) which is widely observed in actual markets and resulting fat-tailness of return distributions and thus likely to underestimate the amount of risk. Monte Carlo method (MC) supposes that the return process obeys a certain time series model such as GARCH model [4] and numerically evaluate a VaR with a Monte Carlo simulation. Since it is a numerical method, MC puts few restrictions on models and theoretically intractable models can be easily evaluated. However, the heavy load of simulation is the shortcoming of MC. Provided that one analyzes the effect of changing weightings of a portfolio on VaR, it may cost enormous number of simulations and may be impossible. To summarize, VC has a fault of underestimating risk, while MC spends much computational resources and has difficulty in the analytical aspect. Hence a method managing both the characteristics of markets such as fat-tailness and the theoretical analysis is needed.

In this paper, we propose a risk evaluating method which preserves the tractable framework of VC method and takes the characteristics of markets into consideration, by the use of Tsallis statistics in statistical physics and its dynamical foundation model, the Beck model. The Tsallis statistics is a generalization of ordinary Boltzmann–Gibbs statistics and is introduced by Tsallis with the motivation of providing a framework of statistical mechanics for far-from-equilibrium systems which often exhibit scaling properties [5]. It is successfully applied to various areas such as fully developed turbulence [6,7], pure-electron plasma [8], cosmic rays [9], economics [10], option pricing [11], and others [12]. The Beck model is introduced by Beck, as a dynamical foundation of Tsallis statistics [13,14]. Although it is originally intended to describe mechanical systems such as turbulence, its basic idea of fluctuating temperature is well-consistent with the heteroskedasticity of markets. Thus, in recent studies, it is employed to elucidate price fluctuations in markets [15,16].

In Section 2, we explain the framework of Tsallis statistics and the Beck model to the extent of requirement of our applications. In Section 3, we perform empirical analysis of daily and high frequency data of S&P500 index to investigate the applicability of the Beck model. In Sections 4 and 5, we deal with two applications of Tsallis statistics and the Beck model: A VaR evaluation method and portfolio risk assessment. In Section 6, we discuss a few points about our applications and Section 7 is devoted to conclusion.

## 2. Theoretical preparations

### 2.1. Tsallis statistics

Tsallis statistics or nonextensive statistical mechanics is a generalization of ordinary Boltzmann–Gibbs statistical mechanics to describe statistical behavior of complex systems. Here we briefly introduce a generalized canonical distribution and its typical realization, $q$-Gaussian distribution [17].

Tsallis introduced a generalized entropy functional [5]

$$S_q[p] = k\frac{1}{q-1}\left(1-\int p(x)^q \mathrm{d}x\right), \quad q > 1, \tag{1}$$

where $q$ is a system dependent parameter and $k$ a positive constant. The parameter $q$ is often called Tsallis' $q$-index. In the limit of $q \to 1$, this functional converges into the ordinary Boltzmann–Gibbs entropy

$$S[p] = k\int p(x)\ln p(x)\mathrm{d}x. \tag{2}$$