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A distributed voltage stability margin for power distribution networks

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Abstract: We consider the problem of characterizing and assessing the voltage stability in power distribution networks. Different from previous formulations, we consider the branch-flow parametrization of the power system state, which is particularly effective for radial networks. Our approach to the voltage stability problem is based on a local, approximate, yet highly accurate characterization of the determinant of the power flow Jacobian. Our determinant approximation allows us to construct a voltage stability index that can be computed in a fully scalable and distributed fashion. We provide an upper bound on the approximation error, and we show how the proposed index outperforms other voltage indices recently proposed in the literature.

Keywords: Voltage stability, power flow Jacobian, power flow solvability.

1. INTRODUCTION

Operators of power distribution grids are facing unprecedented challenges caused by higher and intermittent consumers' demand, driven, among other things, by the penetration of electric mobility. Grid congestion is expected, as the demand gets closer to the hosting capacity of the network. One of the main phenomena that determines the finite power transfer capacity of a distribution grid is *voltage instability* (see the recent discussion in Simpson-Porco et al. 2016). The amount of power that can be transferred to the loads via a distribution feeder is inherently limited by the non-linear physics of the system. In practice, as the grid load approaches this limit, increasingly lower voltages in the feeder are observed, followed by voltage collapse.

From the operational point of view, it is important to be able to identify operating conditions of the grid that are close to voltage collapse, in order to take the appropriate remedial actions. Many different indices have been proposed to quantify the distance of the grid from voltage collapse. Most of them are based on the observation that the Jacobian of the power flow equations becomes singular at the steady state voltage stability limit (see Tamura et al. 1988). For a review of indices based on this approach, we refer to Chebbo et al. (1992) and to Gao et al. (1992).

A geometric interpretation of the phenomena has been developed by Chiang et al. (1990), and starting from Tamura et al. (1983) voltage collapse has been related to the appearance of bifurcations in the solutions of the nonlinear power flow equations.

More recently, semidefinite programming has been proposed as a tool to identify the region where voltage stability is guaranteed (Dvijotham and Turitsyn, 2015). The same region has been also characterized based on applications of fixed-point theorems (see Bolognani and Zampieri 2016 and references therein, and the extensions proposed in Yu et al. 2015 and Wang et al. 2016). Additionally, convex optimization tools have been used to determine sufficient condition for unsolvability (and thus voltage collapse) in Molzahn et al. (2013).

All these works propose *global indices*, in the sense that the knowledge of the entire system state is required at some central location, where the computation is performed. Such a computation typically scales poorly with respect to the grid size, hindering the practical applicability of these methods. Few exception include heuristic indices such as the one proposed in Vu et al. (1999), which can be evaluated by each load based on local measurements.

The methodology that we propose in this paper builds on the aforementioned approach based on the singularity of the power flow Jacobian. Differently from other works, however, we adopt a branch flow model for the power flow equations (Baran and Wu, 1989; Farivar and Low, 2013). This choice gives us a specific advantage, towards three results: first, we reduce the dimensionality of the problem via algebraic manipulation of the Jacobian of such equations; second, we propose an approximation of the Jacobian-based voltage stability margin that is function of only the diagonal elements of the manipulated Jacobian, and is therefore computationally very tractable; finally, we show how such an index can be computed in a fully distributed way, based on purely local measurements at the buses.

The paper is structured in the following way. In Section 2 we recall the branch flow model, while in Section 3 we explain how voltage stability can be assessed based on that model. In Section 4 we propose an approximate voltage stability index and we analyze the quality of the approximation. Finally, in Section 5, we illustrate the result in simulations and we discuss the applicability of this approach to practical grid operation.

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2. POWER DISTRIBUTION NETWORK MODEL

Let $G = (N, E)$ be a directed tree representing a symmetric and balanced radial distribution grid, where each node in $N = \{0, 1, ..., n\}$ represents a bus, and each edge in E represents a line. Note that $|E| = n$. A directed edge in E is denoted by (i, j) and means that i is the parent of j. For each node i, let $\delta(i) \subseteq N$ denote the set of all its children. Node 0 represents the root of the tree and corresponds to the distribution grid substation. For each i but the root 0, let $\pi(i) \in N$ be its unique parent.

We now define the basic variables of interest. For each $(i, j) \in E$ let ℓ_{ij} be the magnitude squared of the complex current from bus i to bus j, and $s_{ij} = p_{ij} + \mathbf{j}q_{ij}$ be the sending-end complex power from bus i to bus j. Let $z_{ij} = r_{ij} + \mathbf{j}x_{ij}$ be the complex impedance on the line (i, j). For each node i, let v_i be the magnitude squared of the complex voltage at bus i, and $s_i = p_i + \mathbf{j}q_i$ be the net complex power demand (load minus generation) at bus i.

In the following, we make use of the compact notation $[x]$, where $x \in \mathbb{R}^n$, to indicate the $n \times n$ matrix that has the elements of x on the diagonal, and zeros everywhere else. Finally, we use the notation $\mathbf{1}_n$ and $\mathbf{0}_n$ for the n-dimensional vectors of all 1's and 0's, respectively.

2.1 Relaxed branch flow model

To model the power distribution network we use the relaxed branch flow equations proposed in Baran and Wu (1989); Farivar and Low (2013) [1]

$$
\begin{aligned}
p_j &= p_{\pi(j)j} - r_{\pi(j)j}\ell_{\pi(j)j} - \sum_{k\in\delta(j)} p_{jk}, \quad \forall j \in N\\
q_j &= q_{\pi(j)j} - x_{\pi(j)j}\ell_{\pi(j)j} - \sum_{k\in\delta(j)} q_{jk}, \quad \forall j \in N\\
v_j &= v_i - 2(r_{ij}p_{ij} + x_{ij}q_{ij}) + (r_{ij}^2 + x_{ij}^2)\ell_{ij}, \ \forall (i,j) \in E\\
v_i\ell_{ij} &= p_{ij}^2 + q_{ij}^2, \quad \forall (i,j) \in E
\end{aligned}
$$

To write these equations in vector form, we first define the vectors p, q, and v, obtained by stacking the scalars p_i, q_i, and v_i, respectively, for $i \in N$. Similarly we define $\overline{p}$, $\overline{q}$, ℓ, r, and x, as the vectors obtained by stacking the scalars p_{ij}, q_{ij}, ℓ_{ij}, r_{ij}, and x_{ij}, respectively, for $(i, j) \in E$.

We define two $(0, 1)$-matrices A^i and A^o, where $A^i \in \mathbb{R}^{n+1\times n}$ is the matrix which selects for each row j the branch (i, j), where $i = \pi(j)$, and $A^o \in \mathbb{R}^{n+1\times n}$ is the matrix which selects for each row i the branches (i, j), where $j \in \delta(i)$. Notice that $A := A^o - A^i$ is the incidence matrix of the graph.

The relaxed branch flow equations in vector form are:

$$
\begin{aligned}
p &= A^i\big(\overline{p} - [r]\ell\big) - A^o\overline{p}\\
q &= A^i\big(\overline{q} - [x]\ell\big) - A^o\overline{q}\\
A^{iT}v &= A^{oT}v - 2\big([r]\overline{p} + [x]\overline{q}\big) + \big([r]^2 + [x]^2\big)\ell\\
\big[A^{oT}v\big]\,\ell &= [\overline{p}]\,\overline{p} + [\overline{q}]\,\overline{q}
\end{aligned}
\tag{1}
$$

We model node 0 as a slack bus, in which v_0 is imposed ($v_0 = 1$ p.u.) and all the other nodes as PQ buses, in which the complex power demand (active and reactive powers) is imposed and does not depend on the bus voltage. Therefore, the quantities $(v_0, p_{1...n}, q_{1...n})$ are to be interpreted as state parameters, and the relaxed branch flow model specifies $4n + 2$ equations in $4n + 2$ state variables, $(\overline{p}, \overline{q}, \ell, v_{1...n}, p_0, q_0)$.

[1] To make the model equations more compact, we adopted the convention $p_{\pi(0)0} = q_{\pi(0)0} = \ell_{\pi(0)0} = r_{\pi(0)0} = x_{\pi(0)0} = 0$.

3. CHARACTERIZATION OF VOLTAGE STABILITY

A *loadability limit* of the power system is a critical operating point (as determined by the nodal power injections) of the grid, where the power transfer reaches a maximum value, after which the relaxed branch flow equations have no solution. There are infinitely many loadability limits, corresponding to different demand configurations. Ideally, the power system will operate far away from these points, with a sufficient safety margin. On the other hand, the *flat voltage solution* (of the power flow equations) is the operating point of the grid where $v = \mathbf{1}_{n+1}$, $p = q = \mathbf{0}_{n+1}$, and $\overline{p} = \overline{q} = \ell = \mathbf{0}_n$. This point is voltage stable and the power system typically operates relatively close to it.

In the following, we recall and formalize the standard reasoning that allows to characterize loadability limits via conditions on the Jacobian of the power flow equations, and we specialize those results for the branch flow model that we have adopted.

3.1 Jacobian of the power flow equations

Based on the discussion at the end of Section 2, consider the two vectors

$$
u = \begin{bmatrix} \overline{p}\\ \overline{q}\\ \ell\\ v_{1...n}\\ p_0\\ q_0 \end{bmatrix} \in \mathbb{R}^{4n+2} \quad \text{and} \quad \xi = \begin{bmatrix} v_0\\ p_{1...n}\\ q_{1...n} \end{bmatrix} \in \mathbb{R}^{2n+1}
$$

corresponding to the state variables and the state parameters, respectively. Then, the relaxed branch flow model (1) can be expressed in an implicit form as

$$\varphi(u, \xi) = \mathbf{0}$$

From a mathematical point of view, a loadability limit corresponds to the maximum of a scalar function $\gamma(\xi)$ (to be interpreted as a measure of the total power transferred to the loads), constrained to the set $\varphi(u, \xi) = \mathbf{0}$ (the physical grid constraints).

$$
\begin{aligned}
\max_{u,\xi} \quad & \gamma(\xi)\\
\text{subject to} \quad & \varphi(u, \xi) = \mathbf{0}
\end{aligned}
$$

From direct application of the KKT optimality conditions, it results that in a loadability limit the power flow Jacobian $\varphi_u = \frac{\partial\varphi}{\partial u}$ becomes singular, i.e., $\det(\varphi_u) = 0$ (for details, see Cutsem and Vournas 1998, Chapter 7). Based on this, we adopt the following standard characterization for voltage stability of the grid.

Definition. (Voltage stability region). The voltage stability region of a power distribution network with one slack bus and n PQ buses, described by the relaxed branch flow model, is the open region surrounding the flat voltage solution where the set of power flow solutions satisfy:

$$\det(\varphi_u) \neq 0 \tag{2}$$