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Lowest order Virtual Element approximation of magnetostatic problems

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Abstract

We give here a simplified presentation of the lowest order Serendipity Virtual Element method, and show its use for the numerical solution of linear magneto-static problems in three dimensions. The method can be applied to very general decompositions of the computational domain (as is natural for Virtual Element Methods) and uses as unknowns the (constant) tangential component of the magnetic field $\boldsymbol{H}$ on each edge, and the vertex values of the Lagrange multiplier p (used to enforce the solenoidality of the magnetic induction $\boldsymbol{B} = \mu\boldsymbol{H}$). In this respect the method can be seen as the natural generalization of the lowest order Edge Finite Element Method (the so-called "first kind Nédélec" elements) to polyhedra of almost arbitrary shape, and as we show on some numerical examples it exhibits very good accuracy (for being a lowest order element) and excellent robustness with respect to distortions.

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1. Introduction

In this paper we introduce a simplified version of the Serendipity Virtual Element Methods (VEMs) presented in [1] and [2] and we show how they can be used for the numerical solution of linear magneto-static problems in the so-called Kikuchi formulation (see e.g. [3]).

Serendipity VEMs are a recent variant of Virtual Element Methods that allow (as is the case of classical Serendipity Finite Elements (FEMs) on quadrilaterals and hexahedra) to eliminate a certain number of degrees of freedom (internal to faces and volumes) without compromising the order of accuracy. In the Virtual Element framework they are

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particularly useful since the original formulations of VEMs (as in [4] or [5]) often use more degrees of freedom than their FEM counterpart (when it exists).

The advantage of VEMs, when it comes to Serendipity variants, is that, contrary to FEMs, they do not use a reference element: an inevitable sacrifice, if you want to be able to deal with very general geometries. Such a sacrifice, that requires additional computations on the current element, has however the advantage of being much more robust with respect to distortions, whereas Serendipity FEMs can lose orders of accuracy already on innocent quadrilaterals that are not parallelograms (as is well known, and has been analyzed e.g. in [6,7]).

Here, as we said, we present a variant of the general theories of [1] and [2], that is specially designed for *lowest order* cases and comes out to be simpler, both for the theoretical presentation and the practical implementation.

Then we apply it to the classical *model* magnetostatic problem, in a *smooth-enough* simply connected bounded domain Ω in $\mathbb{R}^3$:

$$\begin{cases} \text{given}\, \boldsymbol{j} \in H(\text{div};\, \Omega) \quad (\text{with div}\boldsymbol{j} = 0 \text{ in } \Omega), \quad \text{and } \mu = \mu(\boldsymbol{x}) \geq \mu_0 > 0, \\ \text{find } \boldsymbol{H} \in H(\mathbf{curl};\, \Omega) \text{ and } \boldsymbol{B} \in H(\text{div};\, \Omega) \text{ such that:} \\ \mathbf{curl}\boldsymbol{H} = \boldsymbol{j} \text{ and div}\boldsymbol{B} = 0, \text{ with } \boldsymbol{B} = \mu\boldsymbol{H} \text{ in } \Omega \\ \text{with the boundary conditions } \boldsymbol{H} \wedge \boldsymbol{n} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

In particular we shall deal with the variational formulation introduced in [3], that reads

$$\begin{cases} \text{find } \boldsymbol{H} \in H_0(\mathbf{curl};\, \Omega) \text{ and } p \in H_0^1(\Omega) \text{ such that:} \\ \displaystyle\int_\Omega \mathbf{curl}\boldsymbol{H} \cdot \mathbf{curl}\boldsymbol{v}\, \mathrm{d}\Omega + \int_\Omega \nabla p \cdot \mu\boldsymbol{v}\, \mathrm{d}\Omega = \int_\Omega \boldsymbol{j} \cdot \mathbf{curl}\boldsymbol{v}\, \mathrm{d}\Omega \quad \forall \boldsymbol{v} \in H_0(\mathbf{curl};\, \Omega) \\ \displaystyle\int_\Omega \nabla q \cdot \mu\boldsymbol{H}\, \mathrm{d}\Omega = 0 \quad \forall q \in H_0^1(\Omega). \end{cases} \tag{1.2}$$

For many other different approaches to the same problem see e.g. [8–10] and the references therein.

In our discretization, the scalar variable p (Lagrange multiplier for the condition $\text{div}(\mu\boldsymbol{H}) = 0$) will be discretized using only vertex values as degrees of freedom, and the magnetic field $\boldsymbol{H}$ will be discretized using only one degree of freedom (= constant tangential component) per edge. In its turn the current $\boldsymbol{j}$ (here a given quantity) will be discretized by its lowest order *Face Virtual Element interpolant* $\boldsymbol{j}_I$, individuated by its constant normal component on each face.

On tetrahedrons this would correspond to use a piecewise linear scalar for p, a lowest-order Nédélec of the *first kind* for $\boldsymbol{H}$, and a lowest order Raviart–Thomas for $\boldsymbol{j}$: in a sense, nothing new. But already on prisms, pyramids, or hexahedra we start gaining, as we can allow more general geometries and more dramatic distortions, and there are no difficulties in using much more general polyhedrons.

On polyhedrons the present approach could also be seen as being close to previous works on Mimetic Finite Differences (the ancestor of Virtual Elements) like [11] or [12]. Here however the approach is more simple and direct, allowing a thorough analysis of convergence properties. Also the use of an explicit stabilizing term, reminiscent of Hybrid Discontinuous Galerkin methods (see e.g. [13] and the references therein) contributes, in our opinion, to the user-friendliness of the presentation.

A layout of the paper is as follows. The next section will be dedicated to recall the basic notation of functional spaces and differential operators.

Then in Section 3 we will introduce and discuss the two-dimensional VEMs (*nodal* and *edge*) that will be used on the faces of the three-dimensional decompositions. As usual, we will present first the spaces on a single two dimensional element (*local spaces*).

In Section 4 we will finally present our “Simplified Serendipity Spaces” in three dimensions. We first deal with a single element (polyhedron) and then discuss the spaces on a general decomposition.

In Section 5 we will use these spaces to discretize the linear magneto-static problem, and briefly discuss their convergence and the a-priori error analysis.

Finally, in Section 6 we will present some numerical results.

2. Notation

In any dimension, for an integer $s \geq -1$ we will denote by $\mathbb{P}_s$ the space of polynomials of degree $\leq s$. Following a common convention, $\mathbb{P}_{-1} \equiv \{0\}$ and $\mathbb{P}_0 \equiv \mathbb{R}$. Moreover, $\Pi_{s,\mathcal{O}}$ will denote the $L^2(\mathcal{O})$-orthogonal projection onto $\mathbb{P}_s$ (or $(\mathbb{P}_s)^2$, or $(\mathbb{P}_s)^3$). When no confusion can occur, this will be simply denoted by Π_s.