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Finite analytic method for modeling variably saturated flows

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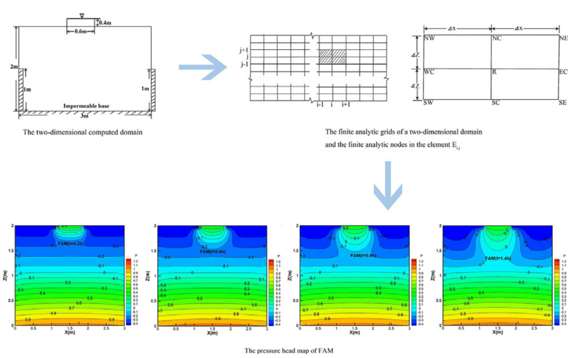
HIGHLIGHTS

- Finite analytic method, which based on Kirchhoff transform, is developed to simulate variably saturated flow.
- The stability of FAM has been proven by a rigorous mathematical analysis.
- Finite analytic method is not only accurate but also efficient, when compared with other numerical methods.

GRAPHICAL ABSTRACT



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ABSTRACT

This paper develops a finite analytic method (FAM) for solving the two-dimensional Richards' equation. The FAM incorporates the analytic solution in local elements to formulate the algebraic representation of the partial differential equation of unsaturated flow so as to effectively control both numerical oscillation and dispersion. The FAM model is then verified using four examples, in which the numerical solutions are compared with analytical solutions, solutions from VSAFT2, and observational data from a field experiment. These numerical experiments show that the method is not only accurate but also efficient, when compared with other numerical methods.

1. Introduction

The process of water movement in the vadose zone is important in agricultural and environmental engineering, soil science, theoretical and applied hydrology as well as water resource management (Melloul and Wollman, 2003; Serrano, 2004; Narula and Gosain, 2013; Bachand et al., 2014; Cleverly et al., 2016; Li et al., 2017). Experiments and mathematical modeling are two effective approaches to study the process. While field experiments (Wang et al., 2011) allow actual observations of the processes in the vadose zone, they are expensive, tedious and time-consuming to implement, and they cannot be used to forecast the processes under different stresses. On the other hand, mathematical modeling is built upon mathematical equations that represent the physics of nature flow processes. The parameters, boundary and initial

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conditions, and source and sink terms can be altered at relative ease to reflect different conditions in the field. Therefore, mathematical modeling is a versatile approach for understanding the flow processes, predicting flow processes, and assessing the uncertainty associated with model predictions due to uncertainty in various model parameters (Yeh et al., 1993a, 1993b; Yeh and Zhang, 1996; Yeh and Simunek, 2002; Wu et al., 2010; Wang et al., 2015; Gholami et al., 2015; Taormina and Chau, 2015, Yeh et al., 2015 book).

Richards' equation has been generally accepted as the appropriate mathematical model for describing flow processes in the vadose zone. However, it is difficult to derive analytical solutions of the equation without making some simplifications due to the nonlinearity of this equation. Numerical solutions to Reichards' equation thus become the alternative. Generally, Richards' equation can be formulated in terms of moisture content, or pressure head, or both moisture content ad pressure head (a mixed formulation, see Yeh et al., 2015 book). The pressure-head based Richards' equation has been widely used to simulate variably saturated flow in geologic materials (e.g., Srivastava and Yeh, 1992; Yeh et al., 1993a, 1993b; Zhang and Yeh, 1997; Li and Yeh, 1999; Hughson and Yeh, 2000; Therrien and Sudicky, 2000; Mayer et al., 2002; Shahrokhabadi et al., 2017). Numerical difficulties associated with solving the pressure-head based Richards' equation exist. For example, when water infiltrates into very dry soils, one may encounter mass-balance error and solution convergence problems (Hao et al., 2005; Lai and Ogden, 2015). Specifically, the numerical treatment of the time derivative term in the pressure-head based equation often yield solutions with poor mass balance. That is, while the time derivative the pressure head in the equation (i.e., $C(h)\,\frac{\partial h}{\partial t}$) and the time derivative of moisture content ($\frac{\partial \theta}{\partial t}$) are mathematically equivalent in the continuous partial differential equation, the numerical approximation of $C(h)\,\frac{\partial h}{\partial t}$ may lead to large computational errors due to high nonlinearity of $C(h)$. These errors cause the mass balance error to grow with increasing time-step size.

To overcome this problem, Rathfelder and Abriola (1994) presented a chord slope approximation to evaluate the soil moisture capacity ($C(h)$). Kavetski et al. (2002) proposed a non-iterative implicit time-stepping scheme with an adaptive truncation error control to solve for pressure head based on Richards' equation.

As for the moisture-content based Richards' equation, the nonlinearity in hydraulic diffusivity $D(\theta)$ and that in hydraulic conductivity $k(\theta)$ are less strong than those in K(h) and C(h) in the pressure-head based equation. In addition, the term $\partial\theta/\partial t$ rather than the term $C(h)\partial h/\partial t$ is used in the moisture-based Richards' equation, and thus the numerical difficulties associated with $C(h)\partial h/\partial t$ do not exist. Hence, the mass balance error in the solution of the moisture-based Richards' equation is small, and this equation is highly suited for modeling infiltration into an initially dry soil. However, the moisture content–based equation also encounters some serious limitations. For example, it cannot simulate water flowing in a saturated zone, since the water diffusivity term becomes infinite and moisture content becomes a constant. It cannot be applied to layered soils since the moisture content is not continuous at the interfaces between different soil types. Likewise, it cannot be used for heterogeneous soils because the moisture is the surrogate of the water energy only under homogeneous soils (see Yeh et al., 2015).

To deal with the discontinuity of soil moisture in moisture content form Richard's equation, Matthews et al. (2004, 2005) extended the moisture content form Richards' equation to layered soil profiles using the Method of Lines (MoL). The MoL method solves the soil moisture discontinuity by considering the flow dynamics within each soil layer separately. Then, the discontinuity is dealt with by means of expressions that satisfy the continuity of flux and pressure head at the interface. Schaudt and Morrill (2002) solved the moisture content form Richards' equation in heterogeneous soils using the continuity of flux and pressure head as boundary conditions. The results show that the method yields excellent water balance and converges as fast as in homogeneous soils.

The mixed form Richards' equation can minimize the mass balance errors by using Picard iteration scheme (Celia et al., 1990; Zha et al., 2017). This form has been implemented in many literature (Yakirevich et al., 1998; Vogel et al., 2001; Yang et al., 2009). But, it is well known that when the pressure head considered as the primary variable, the mixed form Richards' equation performs poorly, especially for problems involving water infiltration into initially dry materials (Forsyth et al., 1995). Moreover, the numerical solution for sharp wetting front can be computationally expensive without adaptive spatial and temporal discretization (Miller et al., 2006). To overcome these problems, the primary variable switching technique proposed (Krabbenhoft, 2007), where the pressure head or soil moisture is used as primary variable relying on the degree of saturation at each node.

As a matter of the fact, the main difficulty in solving the highly nonlinear Richards' equation stems from its hyperbolic characteristic, despite its parabolic form, in terms of the degree of saturation in the solution domain (Ji et al., 2008). To overcome this problem, Ross and Bristow (1990) and Ross (2003) used the Kirchhoff transform to linearize Richards' equation with specific constitutive relationships. By Kirchhoff transform, the nonlinear hyperbolic characteristic of Richards' equation is separated from the parabolic part in each iterative solution stage. Besides, Ross and Bristow (1990) pointed out that the Kirchhoff transformation can be used to eliminate the square of the first derivatives which is in the expanded Richards' equation. In the vicinity of the sharp wetting front, the first derivative requires very small space and time increments. Moreover, using Kirchhoff transformation makes the equation easier to solve numerically than the mixed and pressure-head forms of Richards' equation. It is also different from the saturated-based form as it can be applied to the fully saturated conditions. Therefore, several authors have developed algorithms based on Kirchhoff transformation to solve this form of Richards' equation (Vauclin et al., 1979; Ji et al., 2008; Zhang et al., 2015).

Moreover, the Kirchhoff transform can be a tool to approximate the effective conductivity between the adjacent nodes (Szymkiewicz, 2013). Vauclin et al. (1979) pointed out that the discretized equation after transformation does not require the estimation of the coefficient K at the intermodal calculation points where the solution is unknown. Additionally, because of its integral nature, variations in the U function are much smaller than those in h. This reduces the numerical errors that are with the discretization.

Because of these advantages, Zhang et al. (2015) first introduced finite analytic method (FAM), which applied for Kirchhoff transform, to solve the one-dimensional (1D) unsaturated flow equation. Results showed that FAM can obtain a relatively good mass balance.

FAM was first developed to solve the Navier-Strokes equation (Chen et al., 1981). Solutions utilizing FAM are characterized by an automatic localized upstream shift and an analytic property, which can either eliminate or suppress the difficulty of overall numerical dispersion for large Peclet numbers. To obtain more accurate numerical solutions and better control the global mass balance, Zhang et al. (2016) developed FAM to solve the 1D mixed-form Richard's equation.

In this paper, we applied a more general form of FAM to simulate two-dimensional (2D) variably saturated flow. To evaluate the numerical performance of FAM, several problems typical of variably saturated flow from the literature were simulated using FAM and Variably Saturated Flow and Transport utilizing the Modified Method of Characteristics in 2D (VSAFT2) software (Yeh et al., 1993a, 1993b). This paper is organized as follows: Section 2 describes the Kirchhoff transformation which was used to separate the nonlinear hyperbolic characteristic from the linear parabolic part in each iterative solution stage, it presents the FAM formulation, and it proves the stability of the FAM. Numerical experiments to test the FAM are described in Section 3; Section 4 is the conclusion.

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