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Stationary solutions of discrete and continuous Petri nets with priorities ☆

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ABSTRACT

We study a continuous dynamics for a class of Petri nets which allows the routing at non-free choice places to be determined by priorities rules. We show that this dynamics can be written in terms of policies which identify the bottleneck places. We characterize the stationary solutions, and show that they coincide with the stationary solutions of the discrete dynamics of this class of Petri nets. We provide numerical experiments on a case study of an emergency call center, indicating that pathologies of discrete models (oscillations around a limit different from the stationary limit) vanish by passing to continuous Petri nets.

1. Introduction

Context

The study of continuous analogues of Petri nets dates back to the works of David and Alla [1] and Silva and Colom [2] in 1987. It has given rise to a large scope of research in the field of Petri nets. Whereas classical (discrete) Petri nets belong to the class of discrete event dynamic systems, the circulation of tokens in continuous Petri nets is a continuous phenomenon: tokens are assumed to be fluid, *i.e.*, a transition can fire an infinitesimal quantity of tokens. In this way, the continuous dynamics can be represented by a system of ordinary differential equations or differential inclusions.

Continuous Petri nets are usually introduced as a relaxed approximation of Petri nets, that helps understanding some of the properties of the underlying discrete model, allowing one to overcome the state space explosion that can occur in the latter. The continuous framework can also be seen as a scaling limit of a class of stochastic Petri nets, where the marking M_p of place p in the fluid model is the finite limit of $M_p(N)/N$, with N being a scaling ratio tending to infinity, and where the firing times of transitions follow a Poisson distribution. See [3] for background on scaling limits.

An important effort has been devoted to the comparison between continuous nets and their discrete counterparts. For example, the relationship between reachability of continuous Petri nets and of discrete Petri nets is well understood (see [4]). A recent introduction to continuous models can be found in [5], while a more extensive reference is [6].

In order to evaluate the long-term performance of Petri nets, one has to characterize the stationary or steady states of the Petri nets dynamics. Cohen, Gaubert and Quadrat [7] introduced an approximation of a discrete Petri net by a fluid, piecewise

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affine dynamics with finite delays, and showed that the limit throughput does exist for a class of consistent and free choice Petri nets. In the more recent work of Gaujal and Giua [8], the result is extended to larger classes of Petri nets, and the stationary throughputs are computed as the solutions of a linear program. The results obtained using this fluid approximation hardly apply to the discrete model, up to a remarkable exception identified by Bouillard, Gaujal and Mairesse [9] (bounded Petri nets under total allocation). This reference illustrates the many difficulties that arise from the discrete setting (*e.g.*, some firing sequences may lead to a deadlock).

In the continuous dynamics setting, with time attached to transitions, Recalde and Silva [10] showed that the steady states of free choice Petri nets as well as upper bounds of the throughputs in larger classes of Petri nets can be determined by linear programming. However, in general, the asymptotic throughputs are non-monotone with respect to the initial marking or the firing rates of the transitions [11]. An example of oscillations in infinite time around a steady state is also given in [12].

Contributions

We propose a continuous dynamics of Petri nets where time is attached to places and not to transitions. The main novelty is that it handles a class of Petri nets in which tokens can be routed according to priority rules (Section 2). We initially studied this class in [13] in the discrete setting, motivated by an application to the performance analysis of an emergency call center.

We show that the continuous dynamics can equivalently be expressed in terms of *policies*. A policy is a map associating with every transition one of its upstream places. In this way, the dynamics of the Petri net can be written as an infimum of the dynamics of subnets induced by the different policies. The policies reaching the infimum indicate the places which are bottlenecks in the Petri net. On any time interval in which a fixed policy reaches the infimum, the dynamics reduces to a linear dynamics (Section 3).

We characterize the stationary solutions in terms of the policies of the Petri net. This allows us to set up a correspondence between the (ultimately affine) stationary solutions of the discrete dynamics that were described in [13] and the stationary solutions of the continuous dynamics (Section 4). We also relate the continuous stationary solutions to the initial marking of the Petri net. This relies on restrictive assumptions, in particular the semi-simplicity of a 0 eigenvalue of a matrix associated with a policy.

We finally provide some numerical simulations of the continuous dynamics. We consider a model of emergency call center with two hierarchical levels for handling calls, originating from a real case study (17-18-112 call center in the Paris area) [13]. On this Petri net, numerical experiments illustrate the convergence of the trajectory towards the stationary solution. This exhibits an advantage of the continuous setting in comparison to the discrete one, in which, for certain values of the parameters, the asymptotic throughputs computed by simulations differ from the stationary solutions (Section 5).

Related work

The motivation of this work stems from our previous study [13], in which we addressed the same class of Petri nets with priorities in the discrete setting, and applied it to the performance analysis of an emergency call center. The discrete dynamics is shown there to be given by piecewise affine equations (tropical analogues of rational equations). The idea of modeling priority rules by piecewise affine dynamics originated from Farhi, Goursat and Quadrat [14], who applied it to a special class of road traffic models. In the discrete setting, limit time-periodic behaviors can occur. They may lead to asymptotic throughputs different from the affine stationary solutions of the dynamics, a pathology which motivates our study of a continuous version of the dynamics.

The "continuization" of our dynamics draws inspiration from the original continuous model where time is attached to transitions. In particular, the situation in which the routing of a token at a given place is influenced by the firing times of the output transitions through a race policy has received much attention, see [5]. Here, we address the situation in which the routing is specified by priority or preselection rules which are independent of the processing rates. To do so, it is convenient to attach times to places, instead of attaching firing rates to transitions. We point out in Remark 3 that our model can be reduced to a variant of the standard continuous model [5] in which we allow immediate transitions and require non-trivial routings to occur only at these transitions. A benefit of our presentation is to allow a more transparent comparison between the continuous model and the discrete time piecewise affine models studied in [7,8,13].

The use of the term "policy" refers to the theory of Markov decision processes, owing to the analogy between the discrete time dynamics and the value function of a semi-Markovian decision process. Note that in the context of continuous Petri nets, policies are also known as "configurations", see [11] for an example.

2. Continuous dynamics of Petri nets

2.1. General notation

A Petri net consists of a set $\mathcal{P}$ of places, a set $\mathcal{Q}$ of transitions and a set of arcs $\mathcal{E} \subset (\mathcal{P} \times \mathcal{Q}) \cup (\mathcal{Q} \times \mathcal{P})$. Every arc is given a valuation in $\mathbb{N}$. Each place $p \in \mathcal{P}$ is given an initial marking $M_p^0 \in \mathbb{N}$, which represents the number of tokens initially occurring in the place.