ARTICLE IN PRESS

Physica D ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

A non-perturbative analytic expression of signal amplification factor in stochastic resonance

Asish Kumar Dhara

Variable Energy Cyclotron Centre, 1/AF, Bidhan Nagar, Calcutta-700064, India

HIGHLIGHTS

- We put forward a formalism to evaluate signal amplification factor analytically.
- The formalism takes into account infinite number of perturbation terms.
- The formalism includes the contributions due to infinite number of relaxation modes.
- A closed form analytic expression of signal amplification factor is obtained.
- Only the lowest eigenfunction and Kramers' rate are needed to evaluate the response.

ARTICLE INFO

Article history:
Received 3 August 2016
Received in revised form
7 November 2016
Accepted 10 November 2016
Available online xxxx
Communicated by M. Vergassola

Keywords:
Perturbation series
Fokker–Planck equation
Nonlinear response
Stochastic resonance

ABSTRACT

We put forward a non-perturbative scheme to calculate the response of an overdamped bistable system driven by a Gaussian white noise and perturbed by a weak monochromatic force (signal) analytically. The formalism takes into account infinite number of perturbation terms of a perturbation series with amplitude of the signal as an expansion parameter. The contributions of infinite number of relaxation modes of the stochastic dynamics to the response are also taken into account in this formalism. A closed form analytic expression of the response is obtained. Only the knowledge of the first non-trivial eigenvalue and the lowest eigenfunction of the un-perturbed Fokker–Planck operator are needed to evaluate the response. The response calculated from the derived analytic expression matches fairly well with the numerical results.

1. Introduction

The stochastic resonance [SR] is a noise assisted cooperative phenomenon. The cooperation is manifested as the enhancement of power of a weak monochromatic periodic signal with the help of optimal amount of noise. The system (device) requires to be non-linear and non-equilibrated with its surrounding environment. Thus the non-linear system (taken here as a bistable potential) is interacting with infinite number of degrees of freedom (the environment is taken as Gaussian white noise with noise strength, D) and perturbed by a weak monochromatic periodic force (which we call an input signal with the amplitude, A_0 and the frequency, Ω). The bistable dynamics is therefore stochastic in nature and involves infinite number of relaxation modes. Consequently the response of the system (detected at the output of the device) will also be stochastic. The response of the system is measured as power amplification factor of the input weak signal. Because of the gain of power of the deterministic input signal due to interaction with the random noise, this phenomenon is recognized as noise assisted cooperative response of a non-linear system. The whole system exhibiting SR can then be visualized as a signal processing device to improve the amplification of a weak signal. Thus this phenomenon has been found to be of great importance to modern communication devices (to communicate a signal to a large distance efficiently), to modern detection devices (to detect a feeble signal), applications relating to the enhancement of chemical reaction rates, applications in neurophysiology (especially in the operation of biological motors responsible for movement of muscles), application in novel separation techniques for particles of mesoscopic, micro- and nanoscale sizes (noise induced directed transport) etc. [1–4].

The time evolution of the probability distribution of this non-stationary stochastic process is described by the Fokker–Planck [FP] equation. The exact solution of the Fokker–Planck Equation for this problem is not known. As the signal is weak, one analyzes the system

E-mail address: asishdhara.16@gmail.com.

http://dx.doi.org/10.1016/j.physd.2016.11.002
0167-2789/

Please cite this article in press as: A.K. Dhara, A non-perturbative analytic expression of signal amplification factor in stochastic resonance, Physica D (2016), http://dx.doi.org/10.1016/j.physd.2016.11.002

in terms of a perturbation theory with the amplitude, A_0 of the signal as an expansion parameter. The response of this non-linear device is thus investigated. The first term of the perturbation expansion corresponds to the linear response while the higher order terms in powers of the amplitude of the periodic force correspond to the non-linear responses.

It has been possible to obtain approximate analytical expressions [1,5,6] of the linear response. The non-linear SR responses (signal amplification factor) have been calculated [5] by solving the corresponding Langevin equation numerically without having a recourse to the perturbation theory.

Linear response approximation (considering only the first term of the series) provides the signal amplification factor independent of the amplitude of the input periodic signal while the numerical results show that the response (signal amplification factor) does depend on the amplitude. This suggests that in order to explain amplitude-dependent response one should consider higher order terms of the perturbation series and develop a non-linear response theory.

It is seen from our analysis that the usual method of perturbation approach by truncating the series fails in this case. Violent oscillations emerge due to inclusion of consecutive higher order terms and as a result reasonable finite response is not obtained if we truncate the series. Therefore we put forward a scheme where the truncation of the series in the usual perturbation way is avoided. The formalism proposed by us [7] takes into account

(i) infinite number of perturbation terms

(ii) and the contributions arising due to infinite number of relaxation modes.

The signal amplification factor of a monochromatic periodic signal is considered as a quantifier of stochastic resonance. It is defined as the ratio of the signal power at the output to that at the input and it is given by Eq. (2.9).

The interaction of the input monochromatic signal with the un-perturbed stochastic system generates harmonics of the signal frequency at the output. This is exhibited in Eq. (2.8). This shows that $C_1(x; \mu = 0)$ is the magnitude of the fundamental present in the asymptotic probability distribution. This implies that $\langle \phi_0^\dagger | x | C_1(\mu = 0) \rangle$ (with $\phi_0^\dagger(x) = 1$) in Eq. (2.10) is the effective amplitude of the signal at the output. The quantity, $C_1(x; \mu = 0)$ is expressed as a series with its different orders in Eq. (3.5) where the amplitude A_0 is an expansion parameter. We are taking into account the contributions of infinite number of terms of the perturbation series to the response (signal amplification factor). We calculate the amplitude, $\langle \phi_0^\dagger | x | C_1(\mu = 0) \rangle$ through spectral decomposition of $C_1(x; \mu = 0)$ with respect to the complete set of eigenfunctions of the un-perturbed Fokker–Planck operator.

The amplitude, A_0 and frequency, Ω of the input signal and the noise strength, D constitute the parameters set of this problem. In order to solve this problem analytically we restrict the domain of the parameters, (Ω, D) of this problem where Eqs. (3.6)–(3.8) hold. The input signal is taken to be weak so that the interaction of the signal would not generate the higher harmonics substantially. We assume that the contribution of the fifth harmonic to the response is negligible.

The stochastic dynamics depends on the infinite number of relaxation modes. These modes are characterized by the eigenvalues of the un-perturbed Fokker–Planck operator. The perturbation theory has been developed in [8]. It is seen that each term of the perturbation series depends on infinite number of eigenvalues and eigenfunctions of the un-perturbed Fokker–Planck operator. However, only the lowest eigenfunction and the first non-trivial eigenvalue are known analytically. The analytic expressions of all the other eigenvalues and the eigenfunctions as a function of the noise strength are not known. In our non-linear response theory we construct a 'sum rule' by which the net contribution of infinite number of relaxation modes is obtained analytically. As a result the calculation of the response needs only the knowledge of the lowest eigenfunction and the first non-trivial eigenvalue of the un-perturbed Fokker–Planck operator that is related to Kramers' rate (see the text after Eq. (6.67)).

The perturbation theory constitutes the hierarchical structure of the spectral components of the perturbation with respect to their orders in the perturbation series and the harmonic number. The hierarchy (the index 'k' being the order of the perturbation series) shows that

(a) the spectral component of $(2k+1)$th order of the fundamental, C_1 depends on $(2k-1)$th order of the fundamental, C_1 and $(2k-1)$th order of the third harmonic, C_3 (see Eqs. (3.9)–(3.14))

(b) the $(2k+1)$th order of the third harmonic, C_3 depends on $(2k-1)$th order of the third harmonic, C_3 and $(2k-1)$th order of the fundamental, C_1 (see Eqs. (5.2)–(5.4)).

These two coupled sets of hierarchy are required to be solved to obtain the spectral component of $(2k+1)$th order of the fundamental, C_1, $\langle \phi_m^\dagger | C_1^{(2k+1)}(\mu = 0) \rangle$, which is then employed to obtain the amplitude at the output, $\langle \phi_0^\dagger | x | C_1(\mu = 0) \rangle$ in Eq. (3.15).

In [7] we first solved the hierarchy (a) ignoring the contribution due to third harmonic, obtained the analytical solution of the spectral components of C_1 and call them as ${}^{(1)}\langle \phi_m^\dagger | C_1^{(2k+1)}(\mu = 0) \rangle$. Next we solved the hierarchy (b) approximately and obtain the resonance amplitude, $\langle \phi_0^\dagger | x | C_1(\mu = 0) \rangle$ as a series, where each term of the series can be evaluated with the knowledge of the solution, ${}^{(1)}\langle \phi_m^\dagger | C_1^{(2k+1)}(\mu = 0) \rangle$.

In the present paper we first solve the hierarchy (b) exactly to obtain the spectral component of C_3 in terms of the spectral components of C_1. We next plug this result to the hierarchy (a) and obtain an infinite set of inhomogeneous difference equations involving only the spectral component of C_1 (see Eq. (5.9)). We then solve this reduced hierarchy to obtain the desired spectral component of $(2k+1)$th order of the fundamental, C_1, $\langle \phi_m^\dagger | C_1^{(2k+1)}(\mu = 0) \rangle$.

The paper is organized as follows. We state the problem briefly in Section 2 giving the expressions of the quantities that we are going to derive analytically. As mentioned before, we calculate the amplitude, $\langle \phi_0^\dagger | x | C_1(\mu = 0) \rangle$ through spectral decomposition of $C_1(x; \mu = 0)$. The perturbation theory provides two infinite sets of hierarchy involving the spectral components of C_1 and C_3 which we are going to solve. We outline the procedure to obtain these hierarchies and write them in a compact form in Section 3. As stated before, we reduce these two set of hierarchies into a single set of hierarchy, Eq. (5.9) involving the spectral components of C_1. Eq. (5.9) shows that it involves the quantity, ${}^{(1)}\langle \phi_m^\dagger | C_1^{(2k+1)}(\mu = 0) \rangle$. This is the solution of set of hierarchy, Eqs. (3.9)–(3.11) ignoring the spectral component of C_3, $\langle \phi_m^\dagger | C_3^{(2k-1)}(\mu = 0) \rangle$ (see the text after Eq. (4.6)). This solution has been obtained in [7]. Since this will appear in obtaining the solution of Eq. (5.9) we state them in Section 4 without giving the details of the derivation. The procedure of solving the reduced hierarchy is given in Section 5. Having obtained the spectral components, $\langle \phi_m^\dagger | C_1^{(2k+1)}(\mu = 0) \rangle$ analytically in Section 5 we use them to derive the amplitude, $\langle \phi_0^\dagger | x | C_1(\mu = 0) \rangle$ using Eq. (3.15). The derivation is given in Section 6. It is seen that this procedure gives us a systematic way

Please cite this article in press as: A.K. Dhara, A non-perturbative analytic expression of signal amplification factor in stochastic resonance, Physica D (2016), http://dx.doi.org/10.1016/j.physd.2016.11.002