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Research paper

Evaluation of closure strategies for a periodically-forced Duffing oscillator with slowly modulated frequency subject to Gaussian white noise

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A B S T R A C T

The response of a Duffing oscillator subject to a periodic forcing with slowly and stochastically modulated frequency is analyzed numerically. The results of both moment and cumulant-based stochastic reductions are compared to Monte Carlo simulations. It is shown how the explicit characterization of higher-order central moments of the (Gaussian) noise source and the periodic nature of the forcing enable a reliable reduction strategy providing a faithful description of the mean behavior of stochastic solutions. The reduced model is then used to illustrate how a large noise level and fast frequency drift may combine to sustain a strong response that is normally associated to resonance in the noiseless static case.

1. Introduction

Many time-dependent phenomena in social, physical, and engineering applications, including social exchange [1], traffic flow [2], epidemiology [3], molecular dynamics [4], weather [5], and stock markets [6], are subject to uncertainty, noise, or risk, involving random fluctuations whose impact is often difficult to quantify or predict, but are critical to the evolution process. In this work, we are more specifically interested in processes driven by external forces with a strong periodic component, for example oscillations in ocean surface temperature deviations from seasonal averages (the El Niño effect) [7–9], burst patterns in neuron firing in a nerve axon triggered by small amplitude oscillations in the membrane potential [10], or oscillations in gas flows in nanoscale device manufacturing [4]. Quite often, the periodicity of the phenomenon is also affected by a slow drift in operating or physical conditions, whether by design (e.g. in quasi-static experiments covering a range of settings) or naturally (e.g. changes in the levels of CO_2 and other gases in the atmosphere).

Evolutionary systems subject to noise are typically modeled by stochastic differential equations for which the statistical analysis of direct Monte-Carlo simulations (MCS) is expensive, if not impossible, due to the large number of realizations necessary to obtain meaningful results. Basic mathematical or statistical paradigms can be tested on low-dimensional models reproducing the essential characteristics of the physical problem, and for which MCS are relatively cheap. One such model is the Duffing oscillator with temporal forcing which may have a stochastic component.

The Duffing oscillator problem has a rich history. Its dynamics are well understood in the deterministic and static case [11]. Explicit solutions exist for the unforced case [11,12], as well as for the harmonically forced case [13]. References

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[14,15] vary the forcing amplitude with fixed nonlinear stiffness, which is equivalent to varying the nonlinear stiffness and holding the forcing amplitude constant with an appropriate change of variable. Without the nonlinear stiffness, the Duffing oscillator reduces to a classical linear oscillator, exhibiting oscillations for small damping and resonance when forced near the natural frequency of the oscillator. This resonance is characterized by an increased amplitude response which is often referred to as the resonance response peak. For softening nonlinearity the unforced system has a stable trivial solution and two saddles. The hardening system only has the trivial solution as a stable focus in this regime, but when subjected to negative linear stiffness the hardening system will exhibit bistable behavior with a saddle trivial solution [11, Section 3.3]. Adding forcing causes the resonance peak to tilt toward lower or higher frequencies for softening or hardening systems, and as the magnitude of the nonlinearity increases multiple solutions exist in a range of frequencies below or above the natural frequency. The endpoints of this hysteretic interval are the frequencies at which saddle-node bifurcations occur. For sufficiently large nonlinear stiffness, chaotic or unbounded solutions develop [11, Section 5.4, Section 5.5], but even for more modest stiffness, the choice of initial conditions is very important, as finite-time blow-up can occur for most initial conditions (see the basins of attraction from [14]).

The slow ramping of parameters in the noiseless case has been used extensively in physical experiments to investigate a system's response over a range of parameter values, such as frequencies, typically to save time compared to performing multiple static experiments. The ramp rate must be small enough for the system to relax adiabatically to the long-term state associated with a static experiment at given fixed frequency (quasi-static approximation). Lewis [16] seems to be the first to have investigated the effect of a linear drift in forcing frequency on the behavior of harmonic linear oscillators, and both [17] and [18] independently showed that the solution of the Duffing oscillator can be formulated explicitly in terms of Fresnel integrals. Kevorkian [19] instead considered the modulation of the natural frequency, where the passage through the resonance regime is typically accompanied by a smaller amplitude resonance peak relative to static experiments in the same frequency range, with amplitude oscillations developing after the passage through the resonant frequency [20]. These amplitude oscillations occur only when the damping is small relative to the square root of the ramping rate [21], and are sustained as the damping approaches zero. Slower frequency modulation rates lead to higher amplitude responses since the system can capture more energy in the vicinity of the resonant frequency. Nonlinear stiffness introduces the possibility of blow-up, but otherwise does not modify this heuristic. In particular, it is possible to adjust the ramping rate and maintain large oscillations in the dynamic response beyond the resonance frequency ("auto-resonance") [22,23].

Multiple studies have investigated the effects of noise in the static case, usually with Gaussian white noise as an additional forcing term rather than in the phase [24–26], superposed on the harmonic forcing [26–32]. A common observation is that large white noise tends to weaken nonlinear effects, such as the tilting of the resonance peak. A few studies considered random perturbations directly in the evolution of the phase [15,33,34]. Again, noise has a stabilizing effect on the damped oscillator, but introduces chaos (destroys basins of attraction) in the undamped case [15,35,36]. The additional uniformly distributed phase component in [15] appears to inhibit the Gaussian noise. This is not surprising, since the (wrapped) normal distribution on the circle tends to the uniform distribution for sufficiently large noise.

Here, we address the effect and modelization of noise in the context of slow frequency modulations, with a small fixed damping parameter. This does not seem to have been studied previously. The amplitude response of the Duffing oscillator can be statistically characterized by determining a mean behavior, and possibly variance and covariance with the velocity response or phase, using one of three approaches: (*a*) via Monte Carlo Simulations (MCS) from a large number of (time) realizations of the noise, and computing sample (arithmetic) means and covariances, (*b*) via estimation of a probability density function for each feasible state of the system, as the solution of a (deterministic) Fokker–Planck partial differential equation (PDE), from which ensemble means and covariances of the states can be recovered, or (*c*) via a system of ordinary differential equations (ODE) satisfied by ensemble means and covariances. This system requires closure assumptions, which we discuss below.

Our goals are twofold: (*i*) evaluate moment-based stochastic reduction strategies in approach (*c*) and their viability in providing a faithful account of the amplitude of the solution of the Duffing oscillator under various ramping conditions, and (*ii*) investigate the interaction between the noise level and the frequency ramping rate for a large range of parameter regimes. The approach used here is similar to that adopted in [33], namely strategy (*c*) verified by MCS (*a*), with two major differences: the phase is now drifting and no assumption is made on the noise level, which is possible because of a statistically more suitable characterization of the amplitude response.

2. Background in stochastic modelization and reduction

We consider a non-dimensional Duffing oscillator described by the second-order differential equation

$$\frac{d^2x_t}{dt^2} + \gamma\frac{dx_t}{dt} + (1 - \eta x_t^2)x_t = f(\theta_t), \tag{1}$$

modeling the deviation x_t from the equilibrium position of a mass in a forced nonlinear spring–mass system with (small) positive damping parameter γ and positive stiffness parameter η (softening spring). The forcing term

$$f(\theta_t) := A\sin\theta_t, \tag{2}$$