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Krylov Subspace Methods for Model Order Reduction in Computational Electromagnetics ⋆

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Abstract: This paper presents a model order reduction method via Krylov subspace projection, for applications in the field of computational electromagnetics (CEM). The approach results to be suitable both for SISO and MIMO systems, and is based on the numerically robust Arnoldi procedure. We have studied the model order reduction as the number of inputs and outputs changes, to better understand the behavior of the reduction technique. Relevant CEM examples related to the reduction of finite element method models are presented to validate this methodology, both in the 2D and in the 3D case.

Keywords: Model order reduction (MOR), Krylov subspace method, Arnoldi algorithm, Computational Electromagnetics.

1. INTRODUCTION

In the last decade, computational modeling and simulation has grown as a proper discipline and can be nowadays considered an approach for both the analysis and the synthesis of physical systems that complements theory and experiment. Computer simulations are now performed routinely for several kinds of processes, and, in particular, numerical simulation plays a fundamental role in the study of the complex dynamical phenomena in Electromagnetism (see for example, Lowther (2013)).

In this framework, the model description often comes out from a Finite Element Method (FEM) formulation either in 2D or 3D (Demenko et al. (2014); Sato and Igarashi (2013)), where the input/output/state variables describe the physical relations in the mesh elements of the geometrical discretization of the problem domain (e.g. active and passive structures). The high complexity and level of detail of such a description might be necessary to ensure a certain precision but this directly translates into a high-dimensional state-space representation that can be numerically difficult to treat, because of a high computational and memory cost.

To cope with this problem, the fundamental idea is to derive models of reduced order, capable of giving an accurate description of the real system and at the same time allowing to simplify the design of the controller (Kumar and Nagar (2014); Gugercin and Antoulas (2004); Cenedese et al. (2016); Willcox and Peraire (2002); Benner et al. (2015)). Following this procedure, the system behavior can be described only considering a small group of dominant states, and the main issue is to identify this set to obtain a lower order model that matches (or better approximates) the full order model behavior.

This paper presents a Model Order Reduction (MOR) method via Krylov subspace projection, particularly suitable for strong order reduction of FEM models. The approach is based on the Laurent series expansion of the transfer function of the full order model Σ, to obtain a reduced order model Σ_q that matches the first q expansion coefficients of the original transfer function, so-called *moments* (Salimbahrami and Lohmann (2006)). In order to avoid numerical problem when reducing Σ to Σ_q the classical Arnoldi algorithm is used. The features of this MOR procedure in terms of system behavior are discussed both for Single Input Single Output (SISO) and Multi Input Multi Output (MIMO) systems, and with relevant examples related to 2D and 3D FEM models.

2. PRINCIPLES OF MOR

2.1 System representation

Let $\Sigma(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ be a n-th order continuous-time, MIMO, linear, time-invariant state space model with m inputs and p outputs:

$$\begin{cases} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \quad (1)$$

with state $\mathbf{x}(t) \in \mathbb{R}^n$, input vector $\mathbf{u}(t) \in \mathbb{R}^m$, output vector $\mathbf{y}(t) \in \mathbb{R}^p$, and where $\mathbf{E} \in \mathbb{R}^{n\times n}$, $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, $\mathbf{D} \in \mathbb{R}^{p\times m}$.

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The transfer function of the system is:

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \tag{2}$$

relating the inputs to the outputs by $\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{U}(s)$.

If the number of states n is large (from hundreds to tens of thousands as in FEM models), a computer simulation of the system can be costly (and often unfeasible) in terms of CPU time and memory used. Therefore, when dealing with such high-order models, it is mandatory to try and derive approximated models Σ_q of order $q \ll n$:

$$\begin{cases} \mathbf{E}_q\dot{\mathbf{x}}_q(t) = \mathbf{A}_q\mathbf{x}_q(t) + \mathbf{B}_q\mathbf{u}(t) \\ \mathbf{y}_q(t) = \mathbf{C}_q\mathbf{x}_q(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \tag{3}$$

where $\mathbf{E}_q \in \mathbb{R}^{q\times q},\ \mathbf{A}_q \in \mathbb{R}^{q\times q},\ \mathbf{B}_q \in \mathbb{R}^{q\times m},\ \mathbf{C}_q \in \mathbb{R}^{p\times q}$. In a MOR by projection the system matrices take the form:

$$\mathbf{E}_q = \mathbf{W}^T\mathbf{E}\mathbf{V} \qquad \mathbf{A}_q = \mathbf{W}^T\mathbf{A}\mathbf{V} \tag{4}$$

$$\mathbf{B}_q = \mathbf{W}^T\mathbf{B} \qquad \mathbf{C}_q = \mathbf{C}\mathbf{V} \tag{5}$$

2.2 Moments of the transfer function

The reduction of Σ to Σ_q is expressed as a relationship between the transfer function of the full order system and that of the reduced one. Set $s = \sigma + s_i$ and expand (2) in Laurent series around s_i to obtain:

$$(s\mathbf{E} - \mathbf{A})^{-1} = [\sigma\mathbf{E} + (s_i\mathbf{E} - A)]^{-1} \tag{6}$$

$$= [\sigma(s_i\mathbf{E} - \mathbf{A})^{-1}\mathbf{E} + \mathbf{I}]^{-1}(s_i\mathbf{E} - \mathbf{A})^{-1} \tag{7}$$

$$= \sum_{k=0}^{\infty}(-1)^k\sigma^k[(s_i\mathbf{E} - \mathbf{A})^{-1}\mathbf{E}]^k(s_i\mathbf{E} - \mathbf{A})^{-1} \tag{8}$$

so (2) can be written as:

$$\mathbf{H}(\sigma) = \sum_{k=0}^{\infty}\mathbf{M}_k(s_i)\sigma^k \tag{9}$$

the term $\mathbf{M}_k$ is a $p \times m$ matrix, and is called *moment* of order k of the transfer function:

$$\mathbf{M}_k(s_i) = (-1)^k\mathbf{C}[(s_i\mathbf{E} - \mathbf{A})^{-1}\mathbf{E}]^k(s_i\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \tag{10}$$

This means that, using this technique, we can choose both the accuracy (with a suitable dimension q) and the frequency of the transfer function where moment matching is required. For the most common case of steady state matching ($s_i = 0$), we get:

$$\mathbf{M}_k(0) = (-1)^k\mathbf{C}[-\mathbf{A}^{-1}\mathbf{E}]^k(-\mathbf{A})^{-1}\mathbf{B} \tag{11}$$

$$= -\mathbf{C}[\mathbf{A}^{-1}\mathbf{E}]^k\mathbf{A}^{-1}\mathbf{B} \tag{12}$$

Then the q-order approximated transfer function results:

$$\mathbf{H}_q(\sigma) = \sum_{k=1}^{q}\mathbf{M}_k(s_i)\sigma^k = \mathbf{C}_q(\sigma\mathbf{E}_q - \mathbf{A}_q)^{-1}\mathbf{B}_q \tag{13}$$

This is a general result that can be obtained with a Laurent expansion of the transfer function, and is not referred to more specific reduction technique. Next section will describe how to build the matrices of the reduced system Σ_q.

3. STANDARD BLOCK KRYLOV SUBSPACES METHOD

3.1 Block Krylov subspaces and properties

Let $\mathbf{x} = \mathbf{V}\mathbf{x}_q$ be a change of variable and the matrix $\mathbf{W}$ a suitable matrix that will be described in the following part of the section. With these positions, a reduced order model can be found as:

$$\begin{cases} \mathbf{W}^T\mathbf{E}_q\mathbf{V}\dot{\mathbf{x}}_q(t) = \mathbf{W}^T\mathbf{A}_q\mathbf{V}\mathbf{x}_q(t) + \mathbf{W}^T\mathbf{B}_q\mathbf{u}(t) \\ \mathbf{y}_q(t) = \mathbf{C}_q\mathbf{V}\mathbf{x}_q(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \tag{14}$$

and it can be proved that, if $\mathbf{V},\ \mathbf{W}$ are the basis of a block Krylov subspace referred to the full order system Σ, the moment matching up to the order q is reached.

The *block Krylov subspace* is defined as follows. Let $\mathbf{F} \in \mathbb{R}^{n\times n}$, $\mathbf{G} \in \mathbb{R}^{n\times m}$ be two matrices, then the block Krylov subspace $\mathcal{K}_q(\mathbf{F}, \mathbf{G})$ is defined as

$$\mathcal{K}_q(\mathbf{F}, \mathbf{G}) = colspan\{\mathbf{G}, \mathbf{F}\mathbf{G}, \mathbf{F}^2\mathbf{G}, ..., \mathbf{F}^{q-1}\mathbf{G}\} \tag{15}$$

In particular, we choose the two matrices above such that

$$\mathbf{F} = (s_i\mathbf{E} - \mathbf{A})^{-1}\mathbf{E} \qquad \mathbf{G} = (s_i\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \tag{16}$$

and we write the zero order moment

$$\mathbf{M}_{q,0} = -\mathbf{C}_q\mathbf{A}_q^{-1}\mathbf{B}_q = -\mathbf{C}\mathbf{V}(\mathbf{W}^T\mathbf{A}_q\mathbf{V})^{-1}\mathbf{W}^T\mathbf{B}_q \tag{17}$$

It can be proved that for a particular matrix $\mathbf{V}$ basis of $\mathcal{K}_{q1}(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{B})$ and $\mathbf{W}$ chosen such that $\mathbf{A}_q$ is non-singular, the zero order moment of the reduced model $m_{q,0}$ matches the moment of the full model m_0.

The generalization of this result for a general order q is summarized in the following theorems, which give an idea about one of the main strength of this reduction technique: namely, two choices of the matrices $\mathbf{V}$, $\mathbf{W}$ are available starting from two different Krylov subspaces $\mathcal{K}_{q1}, \mathcal{K}_{q2}$ that depend on particular features of the input and output spaces.

Theorem 1. If the matrix $\mathbf{V}$ used in (4) is a basis of Krylov subspace $\mathcal{K}_{q1}((s_i\mathbf{E} - \mathbf{A})^{-1}\mathbf{E}, (s_i\mathbf{E} - \mathbf{A})^{-1}\mathbf{B})$ with rank q and and $\mathbf{W}$ is chosen such that the matrix $\mathbf{A}_q$ is non-singular, then the first q/m moments (around s_i) of the original and reduced order systems match.

Theorem 2. If the matrix $\mathbf{W}$ used in (4) is a basis of Krylov subspace $\mathcal{K}_{q2}((s_i\mathbf{E}-\mathbf{A})^{-T}\mathbf{E}, (s_i\mathbf{E}-\mathbf{A})^{-T}\mathbf{C}^T)$ with rank q and and $\mathbf{V}$ is chosen such that the matrix $\mathbf{A}_q$ is non-singular, then the first q/p moments (around s_i) of the original and reduced order systems match.

The proof of Theorem 1 (respectively 2) can be obtained by writing moments (9) as linear combinations of the columns of $\mathbf{V}$ (respectively $\mathbf{W}$) basis of subspace $\mathcal{K}_{q1}$ (respectively $\mathcal{K}_{q2}$). For further details see Salimbahrami and Lohmann (2002). These theorems show that the presented approach is able to solve MOR in a flexible way, meaning that we can tackle different order reduction problems with the most suitable subspace. In addition to this, it follows directly from Theorems 1–2 that moment matching is obtained for *any* basis of input or output Krylov subspaces used for order reduction. Depending on the choice of $\mathbf{V}, \mathbf{W}$ we will say respectively *input-Krylov* and *output-Krylov subspace*.

Another important property can be shown by comparing two different reduced models with $\mathbf{V}_1,\ \mathbf{W}_1$ and $\mathbf{V}_2,\ \mathbf{W}_2$, both couple of matrices satisfying theorems 1, 2: