Finite Elements in Analysis and Design 130 (2017) 65–79

Contents lists available at ScienceDirect

Finite Elements in Analysis and Design

journal homepage: www.elsevier.com/locate/finel

A penalty-based finite element framework for couple stress elasticity

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ARTICLE INFO

Keywords:
Consistent couple stress theory
Finite element penalty methods
Size-dependent continuum mechanics
Mixed finite element methods
Mean curvature tensor
Plane strain conditions

ABSTRACT

Length scale based continuum mechanics can account for size dependence - a feature absent in classical theory that is becoming increasingly important in analyzing behavior of matter in micro- and nano-technology. Starting from a foundational mechanics and thermodynamics perspective, recent work has provided a self-consistent formulation to resolve certain issues associated with previous couple stress theory. In the current paper, a penalty-based finite element framework is developed to enable solution of general problems for linear elastic isotropic materials under plane strain conditions within this consistent couple stress theory (C-CST). The penalty parameter is incorporated in the formulation to satisfy continuity requirements, while allowing convergence of the method with only C^0 elements. The efficacy of the finite element analysis is verified by studying three example problems for which closed form solutions are known. Furthermore, a fourth example problem for which a closed form solution is unknown is solved numerically to bring out nontrivial features of couple stress theory. The main objective of this work is to demonstrate that a simple finite element analysis-based framework can be effective in exploring the interesting features of C-CST. Additionally, detailed finite element analysis results provided in this work can be used to benchmark future computational development for size-dependent mechanics theory.

1. Introduction

Classical solid mechanics is based upon length-scale independent measures of deformation, namely, the strain tensor. However, as current technology is evolving towards smaller length scales, one finds that the response is often size-dependent. Consequently, there is a need to develop size-dependent continuum mechanics theory and the corresponding computational mechanics formulations to extend the reach of continuum mechanics and associated tools to describe the material behavior in very small length-scales. One approach for this extension is to revisit the Cauchy assumption that restricts tractions on the surfaces of an elemental continuum volume to those caused exclusively by force-stresses. Instead, as proposed by Voigt [1], couple-stresses can be envisioned as well, along with the corresponding couple-tractions. These theories allow for the existence of couple-stresses that have the dimension of couple (moment) per unit area in addition to the traditional force-stresses. As a consequence, the force-stress tensor is no longer symmetric and the theory allows for material constitutive parameters, which have length-scale dependence. When these characteristic length parameters become insignificant relative to the critical dimensions of the physical problem, the results based on couple stress theory converge to those of the classical continuum theory.

The formulation of such length-scale dependent theories, whose origins go back to the Cosserats [2], has remained a challenge due to some inherent inconsistencies, including the indeterminacy of the couple-stress and force-stress tensors in the initial versions of Couple Stress Theory [3,4]. These unresolved issues have, in turn, encouraged the development of several different size-dependent theories, such as Strain Gradient [5,6] and Micro-Polar theory [7]. These size-dependent formulations are more complicated than the classical theory and closed form solutions have only been possible for a very few select problems in the indeterminate theory [8,9]. However, recent work [10] has contributed new perspectives to resolve the indeterminacy issues in the original Couple Stress Theory. Reference [11] provides the theoretical background for any continuum by using fundamental laws, including the first law of thermodynamics or energy balance equation. In the development of this Consistent Couple Stress Theory (C-CST), the four foundational quantities in mechanics (i.e., force, displacement, couple, rotation) are at the heart of the theory and all individual terms in virtual work, as well as the essential and natural boundary conditions, have clear physical meaning [12]. It also is significant to notice that rigid body mechanics is a special case of C-CST, thus further demonstrating the inner consistency. By analyzing the kinematics of the problem and invoking the fundamental continuum hypothesis, a differential volume is envisioned to resist not only stretches, but also

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http://dx.doi.org/10.1016/j.finel.2016.11.004
Received 8 February 2016; Received in revised form 26 October 2016; Accepted 15 November 2016

Nomenclature

σ_{ji}	Force-stress tensor
μ_{ji}	Couple-stress tensor
μ_i	Couple-stress vector
F_i	Body force per unit volume
ε_{ijk}	Levi-Civita symbol
n_j	Unit vector of surface normal
$t_i^{(n)}$	Force-traction vector on the surface whose unit vector is n_j
$m_i^{(n)}$	Couple-traction vector on the surface whose unit vector is n_j
u_i	Linear displacement vector
e_{ij}	Strain tensor (symmetric)
ω_{ij}	Angular displacement (or rotation) tensor (skew-symmetric)
ω_i	Rotation vector
χ_{ij}	Torsion tensor
κ_{ij}	Mean curvature tensor
κ_i	Mean curvature vector
A_{ijkl}	Constitutive tensor relating symmetric part of force-stress tensor to strain tensor
B_{ij}	Constitutive tensor relating couple-stress vector to mean curvature vector
C_{ijk}	Constitutive tensor relating couple-stress vector or the symmetric part of force-stress tensor to strain tensor and mean curvature vector
δ_{ij}	Kronecker delta
λ	Lamé's first parameter
G	Lamé's second parameter or shear modulus
ν	Poisson's ratio
η	Modulus for isotropic Couple Stress Theory
l	Length scale parameter
E	Young's modulus
E_{TOT}	Total strain energy
E_{CL}	Strain energy contribution due to force-stress
E_{CS}	Energy contribution due to couple-stress
E_P	Energy contribution due to penalty parameter
P	Penalty parameter
$\varnothing$	Null Set

bending deformation. This leads to the energy conjugate pair, consisting of the skew-symmetric mean curvature and couple-stress tensors that need to be accounted for in the energetics. Therefore, the potential energy density for an elastic solid due to isothermal mechanical deformation (henceforth, referred to as elastic energy density) depends, not only on the strain tensor, but also the mean curvature tensor that is defined as the skew-symmetric part of the gradient of the rotation vector. Interestingly, the couple-stress and mean curvature tensors are pseudo (axial) tensors, while their corresponding duals are represented by true (polar) vectors.

This fully determinate size-dependent continuum theory forms the foundation for the penalty-based finite element analysis (FEA) framework proposed in the current work. The penalty FEA framework is validated for four representative problems in plane strain C-CST using either closed-form solutions or existing converged solutions from other numerical methods, when closed-form solutions are unavailable. The utility of the current work can be understood with the following example. With multiple theories already existing in the size-dependent polar continuum mechanics literature [4–7], the emergence of C-CST has created a debate in the research community [13–18]. In this context, the present penalty FEA framework can serve mechanics researchers and engineers seeking a better understanding of C-CST via numerical simulations within the scope of their problems of interest. Furthermore, the numerical results provided here can also serve to benchmark new development on the computational mechanics front for size-dependent theories.

A pure displacement-based FEA implementation of Couple Stress Theory would require the continuity of the angular displacement (or rotation) over the problem domain, hence a C^1 continuity on the displacement field is desired. Such C^1 elements based on Hermite shape functions have been developed for both 2-dimensional [19] and 3-dimensional [20] boundary value problems. Owing to the challenges in the developing C^1 continuous basis functions, a more popular approach is to use shape functions with less restrictive regularity conditions imposed and enforce the desired continuity on the higher derivatives in a weak sense. In this context, some common approaches include Discontinuous Galerkin (DG), Continuous/Discontinuous Galerkin (C/DG), Hybrid and Mixed Finite Element Methods.

In the DG approach, the approximations for the solution field are not continuous across elements and the inter-element continuity in the solution field and higher order derivatives are weakly imposed via penalizing the jumps of these functions and their derivatives across the element boundaries using stabilization parameters [21]. This approach has been analyzed [22] and demonstrated for strain gradient-dependent damage problems [23]. In the C/DG approach, the DG approach is modified to include C^0 continuous solution fields and penalize the jumps in the higher order derivatives using stabilizing parameters. The C/DG does not require the assumption of an independent higher order gradient field. The C/DG theory has been analyzed and demonstrated [24,25] in higher order theories for Toupin-Mindlin shear layer [26,27] and nanocrystalline microstructures with internal grain boundaries [28]. An interesting feature of the C/DG approach [25] is that it can respect the discontinuity in the higher order derivatives of the solution field at the material grain interfaces and junctions, which is a physically permissible phenomenon as per the fundamental model [28], while simultaneously trying to reduce the discontinuity in the solution field at the grain interfaces or its derivatives at the internal edges of the finite elements stemming from the numerical approximation using C^0 basis functions.

Mixed and Hybrid Finite Element Methods can be categorized under multi-field variational formulations [29]. A hybrid finite element is characterized by additional variables that are defined on the interior element edges, whereas in the mixed finite element method multiple fields are defined on a finite element [30]. In both these classes of problems, a Lagrange multiplier method is commonly used to convert the constrained minimization problem to a saddle-point problem. Problems in Strain Gradient Theory based elasticity and plasticity have been solved via C^0 continuous elements by using Lagrange multipliers to weakly impose the C^1 continuity [31–33]. In addition, FE methods have been developed for planar problems, based on the indeterminate Couple Stress Theory, using mixed finite element formulations [34,35]. In recent work, a Lagrange multiplier finite element method has been developed for problems in isotropic elastostatics in C-CST [36]. A finite element method, where additional non-conforming displacement fields are defined in addition to a C^0 continuous displacement field, have also been shown to weakly enforce inter-element continuity for indeterminate Couple Stress Theory [37]. Another technique that can be classified as a Mixed element method is that of the penalty-based finite element. The penalty method can be advantageous when the indefiniteness of the system matrix associated with a Lagrange multiplier-based mixed formulation is undesired or when elements with lower number of degrees of freedom are desired. Here a positive definite function of constraint violation, scaled by a penalty term, can be appended to the potential energy function that will be minimized [38]. Weak C^1 continuity has been shown to be imposed by penalizing the continuity constraint violation for the indeterminate Couple Stress Theory [39,40]. The basic principles of penalty-based FEA can be found