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An uncoupled higher-order beam theory and its finite element implementation

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ABSTRACT

A beam problem is though classical but not theoretically settled down at present. Different from the previous work, the current study starts with defining the generalized displacements. Together with the assumptions and the shear stress free condition, the axial displacement is first mathematically expanded in two terms and then decomposed into an orthogonal form in terms of the generalized displacements. The generalized stresses are accordingly defined, and the uncoupled constitutive relations are derived for beam problems after the generalized strains are properly measured. The principle of virtual work is proposed and the variationally consistent higher-order beam theory is eventually established, which can reduce to the variationally asymptotic lower-order beam theory. With these fundamentals, the finite element method is finally formulated as easy as for a three-dimensional elastic problem, and applied to typical problems. The results show that the higher-order beam element can capture the effect of the clamped end and the load jump via smoothly modeling the warping of cross section by using a locally refined mesh while accurately modeling the deflection. With the current framework, modern beam structures accounting for the effect of nonlocal elasticity, small scales and material heterogeneities can be readily solved.

1. Introduction

Beam problems are classical in the mechanics field, but still of great importance in modern scientific researches and engineering applications. For example, the laminate beams [1], the functionally graded material beams [2], the piezoelectric beams [3], the shape memory alloy beams [4], and the beams considering nonlocal elasticity [5], gradient elasticity [6] and couple stresses [7], or small-sized beams with small scale effects [8] have been widely studied in recent years. As the fundamental outcome, massive research developments have been reported since the advent of Euler–Bernoulli beam theory (EBT) [9]. However, some issues are still pending for beam problems.

The first issue is the concept of generalized displacement. As we know, for a three-dimensional elastic problem, the displacement vector is essential, based on which strains are measured. However, the generalized displacement is not definitely defined [10] for beam problems. It is no doubt that, as a flexible structure, the deflection is certainly the primary displacement, but how to define deflection pertaining to the vertical displacement (i.e. $v(x, y)$ for a plane problem) is controversial. It turns out that the deflection is mostly defined as vertical displacement at the neutral line [11,12] while sometimes defined as the average of vertical displacement over the cross section [13]. Owing to the assumption that the lateral normal strain vanishes for a beam problem, the two definitions are in effect identical, and hence this issue appears to be trivial. The other generalized displacement in beam problems is the rotation of cross section, which plays an indispensable role either for shear deformable theory or for the classical theory. The fact is that the beam theories can be studied in a unified way if enough attention is paid to this concept. In previous researches, the rotation is mostly defined as the one at the neutral line [11,12] while sometimes defined as the average over the cross section [13,14], even others [15,16]. Unlike the deflection, various rotations have different physical meanings, giving rise to different constraints at the clamped end [12,14]. In addition, to establish a perfect beam theory, it is hopeful to conceptually conform the generalized displacements with the generalized stresses.

The second issue is about generalized stress. It has been well known that moment and shear force are the two generalized stresses in beam theories, as well as in the theory of strength of materials, corresponding to rotation and deflection as their generalized displacements. However, the situation changes for the beam theories with a higher-order mode of axial displacements. Although many researchers tried to define the higher-order generalized stresses [17], they are by no means the "moment" or "shear force" from the dimensional viewpoint. Particu-

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larly, those defined higher-order stresses [17] are really not higher-order (small) quantities which can be ignored when deriving the lower-order beam theory.

The third issue is about generalized strain. As we know, the generalized strains measured by the generalized displacements should be work conjugated with the generalized stresses. So, this issue is closely correlated with the first two issues. Unfortunately, due to the inadequacy of this issue, some confusions appear even in the recent researches [18]. This is just our motivation to re-evaluate the beam theories.

Were the three issues well settled, the beam theory could be systematically studied, leading to the governing ordinary differential equations (higher-order or lower-order) and corresponding boundary conditions, as well as the finite element formulations.

However, the researches on higher-order beam theories are not along this guideline, which can be viewed from the corresponding finite element implementation. As early as in 1988, Heyliger and Reddy [19] proposed a higher-order beam finite element for bending and vibration problems. Because this element was so devised as to follow their theoretical work (e.g. see [17]) in which the constitutive relations were not established for beam problems, the stiffness matrix was obtained block by block in a tedious way. Eisenberger [20] proposed an exact higher-order beam element in which the terms in the stiffness matrix were the holding actions at the end of the beam element in the higher-order beam theory when the beam was so deformed that the desired degree of freedom (DOF) was unity while all other DOFs were restrained. Without use of shape functions for the element, this is however not a general and acceptable practice for the finite element method. Murthy et al. [21] proposed a refined higher-order finite element and then applied it to asymmetric composite beams in which the shape functions were exactly derived by satisfying the governing equations of higher-order beam theory, and the stiffness was expressed in a matrix form. However, it appears troublesome when obtaining the shape functions and the strain-displacement matrix. Although these exact or refined elements are very successful for lower-order beam theories such as the Timoshenko beam theory (TBT) [22–24], they are not appropriate for higher-order beam theories, especially when extended to plate problems which are governed by partial differential equations.

In this paper, a mechanics manner is proposed to study beam problems by replacing the previous engineering manner. That is, a beam problem is completely stated as the approximation of a three-dimensional or two-dimensional elastic problem with some assumptions and under some conditions. To this end, the paper is outlined as follows. In Section 2, the fundamental of a beam problem is briefly summarized, including the problem description, the kinematics, the concept of generalized stresses and strains, and the principle of virtual work. In Section 3, the higher-order beam theory is first derived, including the equilibrium equations and boundary conditions, and governing ordinary differential equations, and then reduces to the lower-order beam theory. In Section 4, the finite element method for beam problems is formulated for the present higher-order beam theory. Typical examples are numerically solved and then discussed in Section 5 to validate the proposed higher-order beam element. The concluding remarks are finally given in Section 6.

2. Fundamentals

2.1. Description of a beam problem in the manner of plane elasticity

For the explanation reason, our focus is on a straight beam with rectangular cross section. As shown in Fig. 1, the beam structure with unit width is located in x-y plane with thickness h and length L. As a plane stress problem, the elasticity theory is to solve the displacement field (u, v) varying with field point (x, y). In contrast, the corresponding beam theory is to solve the deflection w varying with length coordinate x.

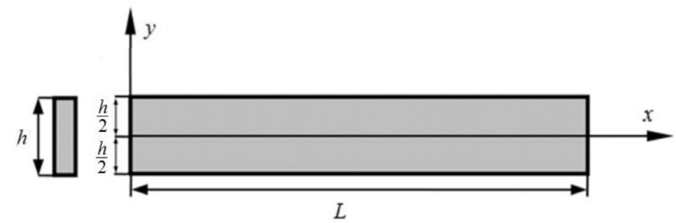

Fig. 1. A beam structure with rectangular section.

In the manner of beam, due to the small thickness-to-length ratio (i.e. h/L), the following two assumptions are often adopted.

Assumption 1. The lateral normal strain ε_y vanishes identically, i.e.

$$\varepsilon_y = 0 \tag{1}$$

which implies that

$$\frac{\partial v(x,y)}{\partial y} = 0 \tag{2}$$

Assumption 2. The stress components for a plane problem satisfy [25]

$$\sigma_y \ll \tau_{xy} \ll \sigma_x \tag{3}$$

where σ_x, σ_y and τ_{xy} are, respectively, the longitudinal normal stress, lateral normal stress and the shear stress.

Thus, ignoring σ_y, Hooke's law yields

$$\begin{cases} \sigma_x = E\varepsilon_x \\ \tau_{xy} = G\gamma_{xy} \end{cases} \tag{4}$$

where E and G are, respectively, Young's modulus and shear modulus. ε_x and γ_{xy} are, respectively, the longitudinal normal strain and shear strain with strain measures being

$$\begin{cases} \varepsilon_x = \frac{\partial u}{\partial x} \\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{cases} \tag{5}$$

2.2. Kinematics of a beam

According to the transverse displacement v(x, y), the deflection of beam is defined as

$$w(x) = (1/A)\int_A v(x,y)dA \tag{6}$$

where A is the area of cross section. Eq. (6) implies that the current deflection is in the average sense over the cross section.

In engineering, the deflection w is mostly defined as

$$w(x) = v(x,0) \tag{7}$$

With Eq. (1), we also have

$$w(x) = v(x,y) = v(x,h/2) \tag{8}$$

Considering Eq. (1), the definition by Eq. (6) satisfies Eqs. (7) and (8) as well. Thus, with Assumption 1, the two definitions of deflection are in effect identical.

According to axial displacement u(x, y), the rotation of cross section for a beam is defined also in the average sense over the cross section as

$$\phi(x) = (1/I)\int_A y u(x,y)dA \tag{9}$$

where I, the second-order moment of inertia, is

$$I = \int_A y^2 dA \tag{10}$$

w in Eq. (6) and ϕ in Eq. (9) are called generalized displacements of beam.