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Optimal network design for synchronization of coupled oscillators☆

Mahyar Fazlyab [a], Florian Dörfler [b], Victor M. Preciado [a]

[a] *University of Pennsylvania, Department of Electrical and Systems Engineering, United States*
[b] *ETH Zurich, Automatic Control Laboratory, Switzerland*

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ABSTRACT

This paper studies the problem of designing networks of nonidentical coupled oscillators in order to achieve a desired level of *phase cohesiveness*, defined as the maximum asymptotic phase difference across the edges of the network. In particular, we consider the following two design problems: (i) the *nodal-frequency design problem*, in which we tune the natural frequencies of the oscillators given the topology of the network, and (ii) the (robust) edge-weight design problem, in which we design the edge weights assuming that the natural frequencies are given (or belong to a given convex uncertainty set). For both problems, we optimize an objective function of the design variables while considering a desired level of phase cohesiveness as our design constraint. This constraint defines a convex set in the nodal-frequency design problem. In contrast, in the edge-weight design problem, the phase cohesiveness constraint yields a non-convex set, unless the underlying network is either a tree or an arbitrary graph with identical edge weights. We then propose a convex semidefinite relaxation to approximately solve the (non-convex) edge-weight design problem for general (possibly cyclic) networks with nonidentical edge weights. We illustrate the applicability of our results by analyzing several network design problems of practical interest, such as power redispatch in power grids, sparse network design, (robust) network design for distributed wireless analog clocks, and the detection of edges leading to the Braess' paradox in power grids.

1. Introduction

The analysis of synchronization in networks of coupled oscillators is one of the most fundamental problems in the field of networked dynamical systems. Networks of coupled oscillators present a rich dynamic behavior, as reported in the vast literature on this topic; see, for example, Dörfler and Bullo (2014) and references therein. Many complex artificial and natural systems can be modeled as networks of coupled oscillators, such as pacemaker cells in the heart, neurons in the brain, clocks in computing networks, mobile sensor networks, and power grids. Considerable research in this field has been focused on studying the effect of network structure, coupling strengths, and nodal dynamics on the ability of a network of oscillators to synchronize (di Bernardo, Garofalo, & Sorrentino, 2007; Menck, Heitzig, Kurths, & Schellnhuber, 2014; Sorrentino, Di Bernardo, & Garofalo, 2007). Various metrics have been proposed in the literature to quantify and optimize the synchronization performance. A broad class of these metrics focuses on the transient response, such as the ability of the network to resynchronize after perturbations (Donetti, Hurtado, & Munoz, 2005; Kempton, Herrmann, & di Bernardo, 2015; Motter, Zhou, & Kurths, 2005; Pecora & Carroll, 1998). In this context, synchronizability can be characterized by either the required effort to synchronize the network (Sjödin, Bamieh, & Gayme, 2014), the speed of convergence to the synchronization manifold (Fardad, Lin, & Jovanović, 2014b; Xiao & Boyd, 2004), or the range of coupling values for which a network with uniform coupling strengths would synchronize (Pecora & Carroll, 1998). Using the master stability framework, proposed in Pecora and Carroll (1998), it was shown that the Laplacian algebraic connectivity and the Laplacian eigenratio are two network-dependent measures able to capture the synchronizability of a network of *identical* coupled oscillators. Based on this connection, we find in the literature several works aiming to optimize the synchronizability of a network of identical coupled oscillators using the Laplacian matrix (Clark, Alomair, Bushnell, & Poovendran, 2014; Donetti et al., 2005; Fardad, Lin, & Jovanovic, 2014a; Kempton et al., 2015; Motter, Myers, Anghel, & Nishikawa, 2013; Motter et al., 2005; Mousavi, Somarakis, & Motee, 2016; Nishikawa & Motter, 2006; Pecora & Carroll, 1998; Rad, Jalili, & Hasler, 2008; Siami & Motee, 2016; Skardal & Arenas, 2015).

In Dörfler, Chertkov, and Bullo (2013), the concept of *phase cohesiveness*, defined as the maximum steady-state phase difference

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E-mail addresses: mahyarfa@seas.upenn.edu (M. Fazlyab), dorfler@ethz.ch (F. Dörfler), preciado@seas.upenn.edu (V.M. Preciado).

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across the edges of a network, was proposed as a synchronization metric in networks of *nonidentical* coupled oscillators. This metric explicitly accounts for the simultaneous effect of the network topology, the coupling strengths, and the nodal dynamics on the local stability of the synchronous solution. This paper adopts phase cohesiveness as a synchronization measure in order to develop an optimization framework for designing the parameters of a network of coupled oscillators. As described in Section 2, the oscillators in our network are modeled using the *swing equation*, widely used in the analysis of power grids (Bergen & Hill, 1981). Specifically, we address the following design problems:

(1) *Design of natural frequencies*: In this problem, we assume that the network structure and the coupling strengths are given. The network designer is able to tune the natural frequencies of each oscillator by incurring a cost. The objective is to minimize the total tuning cost while guaranteeing a desired level of phase cohesiveness.
(2) *Design of link weights*: In this second problem, we assume that the natural frequencies of the oscillators belong to a given polyhedral uncertainty set. The network designer is able to tune the edge weights by incurring a cost. The goal is to minimize the total tuning cost while guaranteeing a desired level of phase cohesiveness for all possible realizations of the natural frequencies in the uncertainty set.

The framework herein proposed can be used in a wide range of practical applications, namely, prevention of cascading failures in power grids (Linnemann, Echternacht, Breuer, & Moser, 2011), optimal design of electrical infrastructure upgrades, sparsity promoting network design (Dhingra, Lin, Fardad, & Jovanović, 2012; Lin, Fardad, & Jovanović, 2012; Siami & Motee, 2015), and detection of links inducing the Braess' paradox (i.e., the counter-intuitive phenomenon of losing synchronization as the result of adding new edges (Witthaut & Timme, 2012)). We will discuss some of these applications in Section 5.

The rest of the paper is organized as follows. Section 2 provides some background on the synchronization problem. Section 3 develops an optimization framework to solve the frequency design problem. The (robust) weight design problem is solved in Section 4. Illustrative examples are presented in Section 5. Concluding remarks are drawn in Section 6.

Notation: Let $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_{++}$ be the set of real, nonnegative, and strictly positive numbers. Let $\mathbf{1}_n$ and $\mathbf{0}_n$ be the n-dimensional vectors of unit and zero entries. The set $\{1,\ldots,n\}$ is denoted by $[n]$. The infinity norm of $\mathbf{x}\in\mathbb{R}^n$ is denoted as $\|\mathbf{x}\|_\infty=\max_i|x_i|$, the ℓ_1 norm as $\|\mathbf{x}\|_1=\sum_{i=1}^n|x_i|$, and the ℓ_0 norm as $\|\mathbf{x}\|_0=\mathrm{card}(\{i\in[n]:x_i\neq 0\})$, where $\mathrm{card}(\cdot)$ denotes the cardinality of a set. For $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$, the inequality $\mathbf{x}\leq\mathbf{y}$ is component-wise. We denote by $\mathbb{S}^{n\times n}$ the set of $n\times n$ real, symmetric matrices. For square matrices A and B, we write $A\succeq B$ if and only if $A-B$ is positive semidefinite.

Elements of algebraic graph theory: A graph is defined as $G=(\mathcal{V},\mathcal{E})$, where $\mathcal{V}$ is a set of n nodes and $\mathcal{E}$ is a set of m undirected edges. We assume that the graph is connected and has no self-loops. We consider graphs with weights associated to both edges and nodes. We denote the weight of an edge $e=\{i,j\}\in\mathcal{E}$ as $w_e=w_{ij}$. The *weighted adjacency matrix* of an undirected graph G, denoted by $A=[a_{ij}]$, is an $n\times n$ symmetric matrix defined entry-wise as $a_{ij}=w_{ij}$ if $\{i,j\}\in\mathcal{E}$, and $a_{ij}=0$, otherwise. The *weighted Laplacian matrix* of G is defined as $L=\mathrm{diag}\,(A\mathbf{1}_n)-A$. For an edge $e=\{i,j\}\in\mathcal{E}$, we define $\mathbf{b}_e\in\mathbb{R}^n$ with $b_{e,i}=1,\ b_{e,j}=-1$ (or $b_{e,i}=-1,\ b_{e,j}=1$) and all other entries equal to zero. The incidence matrix $B\in\mathbb{R}^{n\times m}$ is the matrix with e th column $\mathbf{b}_e$. For a weighted graph, we define the edge-weight vector $\mathbf{w}=(w_1,\ldots,w_m)^\top$, where w_e is the weight of the edge labeled e. The Laplacian matrix of the weighted graph can be written as $L(\mathbf{w})=B\mathrm{diag}\,(\mathbf{w})\,B^\top$. The *Moore–Penrose pseudoinverse* of the Laplacian is defined as $L(\mathbf{w})^\dagger=\left(L(\mathbf{w})+\frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top\right)^{-1}-\frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top$. For any connected graph with n vertices, the identity $L(\mathbf{w})L(\mathbf{w})^\dagger=L(\mathbf{w})^\dagger L(\mathbf{w})=I_n-\frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top$ holds.

2. Synchronization in networks of heterogeneous oscillators

Consider the partition $\{\mathcal{V}_1,\mathcal{V}_2\}$ of a set of n nodes in a connected, weighted, undirected graph $G(\mathcal{V},\mathcal{E})$. The state of each node $i\in\mathcal{V}$ is represented by an angular position $\theta_i\in\mathbb{R}$ whose dynamics is described by the following set of differential equations:

$$m_i\ddot{\theta}_i+d_i\dot{\theta}_i=\omega_i-\sum_{j=1}^n a_{ij}\sin\left(\theta_i-\theta_j\right),\ i\in\mathcal{V}_1, \tag{1a}$$

$$d_i\dot{\theta}_i=\omega_i-\sum_{j=1}^n a_{ij}\sin\left(\theta_i-\theta_j\right),\ i\in\mathcal{V}_2. \tag{1b}$$

Here, $\mathcal{V}_1$ is the subset of oscillators following a second-order dynamics with inertia $m_i>0$ and damping coefficient $d_i>0$, and $\mathcal{V}_2$ is the subset of oscillators with a first-order dynamics; $\omega_i\in\mathbb{R}$ is the natural frequency of the i th oscillator (which corresponds to power generation/consumption in generator/load buses), and $a_{ij}\geq 0$ is the (ij)-th entry of the weighted adjacency matrix of $G(\mathcal{V},\mathcal{E})$. The dynamics in (1) represents the swing dynamics for a structure-preserving lossless power network with constant voltage magnitudes at the buses (Bergen & Hill, 1981). This dynamics can be written in matrix form as

$$M\ddot{\boldsymbol{\theta}}+D\dot{\boldsymbol{\theta}}=\mathbf{f}(\boldsymbol{\theta})=\boldsymbol{\omega}-BW\mathbf{sin}(B^\top\boldsymbol{\theta}), \tag{2}$$

where $\boldsymbol{\theta}=(\theta_1,\ldots,\theta_n)^\top$, $\boldsymbol{\omega}=(\omega_1,\ldots,\omega_n)^\top$,
$M=\mathrm{diag}(\{m_i\}_{i\in\mathcal{V}_1},\mathbf{0}_{|\mathcal{V}_2|})$ is the diagonal matrix of inertias, $D=\mathrm{diag}(\{d_i\}_{i\in\mathcal{V}})$ is the diagonal matrix of damping coefficients, B is the incidence matrix of G, $W=\mathrm{diag}\,(\mathbf{w})$ is the diagonal matrix of edge weights, and $\mathbf{w}=(w_1,\ldots,w_m)^\top$ where $w_e>0$ is the weight of the e th edge in the graph. The special case $\mathcal{V}_1=\emptyset$ and $d_i=1$ corresponds to the classical Kuramoto model (Acebrón, Bonilla, Vicente, Ritort, & Spigler, 2005). The following definition characterizes the notion of synchronization for (2).

Definition 2.1. A solution $\boldsymbol{\theta}(t)$ to the coupled oscillator model (1) is said to be *frequency-synchronized* if $\lim_{t\to\infty}|\theta_i(t)-\theta_j(t)|(\bmod\ 2\pi)=\varphi_{ij}^\star$, for all $\{i,j\}\in\mathcal{E}$ and some $\varphi_{ij}^\star\in[0,2\pi)$. Furthermore, if $\varphi_{ij}^\star=0$ for all $\{i,j\}\in\mathcal{E}$, the solution is said to be *phase-synchronized*.

Phase synchronization can only be achieved if all the natural frequencies are identical. In contrast, if the natural frequencies are not all identical, the network can only achieve frequency synchronization. For a frequency-synchronized solution, the angular velocities of the oscillators converge towards a common asymptotic frequency given by $\omega_s=\sum_{i=1}^n\omega_i/\sum_{i=1}^n d_i$ (Dörfler and Bullo, 2011, § 5.2). Thus, the frequency-synchronized solution satisfies $\lim_{t\to\infty}(\boldsymbol{\theta}(t)-\boldsymbol{\theta}_s(t))(\bmod\ 2\pi)=\mathbf{0}_n$, where $\boldsymbol{\theta}_s(t)=(\omega_s t)\mathbf{1}_n+\boldsymbol{\theta}^\star$ for some $\boldsymbol{\theta}^\star\in\mathbb{R}^n$ such that $M\ddot{\boldsymbol{\theta}}_s+D\dot{\boldsymbol{\theta}}_s=\mathbf{f}(\boldsymbol{\theta}_s)$. It then follows from Definition 2.1 that a frequency-synchronized solution $\boldsymbol{\theta}(t)$ satisfies $\lim_{t\to\infty}|\theta_i(t)-\theta_j(t)|=|\theta_i^\star-\theta_j^\star|,\ \forall\{i,j\}\in\mathcal{E}$.

Definition 2.2. For any frequency-synchronized solution $\boldsymbol{\theta}_s(t)=(\omega_s t)\mathbf{1}_n+\boldsymbol{\theta}^\star$ of (2), the corresponding phase cohesiveness is defined as

$$\begin{aligned}\varphi(B,\mathbf{w},\boldsymbol{\omega})&=\max_{\{i,j\}\in\mathcal{E}}\lim_{t\to\infty}|\theta_i(t)-\theta_j(t)|(\bmod 2\pi)\\&=\|B^\top\boldsymbol{\theta}^\star\|_\infty\,(\bmod 2\pi).\end{aligned} \tag{3}$$

Without loss of generality, we can assume that $\omega_s=0$ by introducing a rotational reference frame in which $\omega_s=0$. It then follows that $\boldsymbol{\theta}_s(t)=\boldsymbol{\theta}^\star$ and $\mathbf{0}_n=M\ddot{\boldsymbol{\theta}}_s+D\dot{\boldsymbol{\theta}}_s=\mathbf{f}(\boldsymbol{\theta}_s)=\mathbf{f}(\boldsymbol{\theta}^\star)$, i.e., the frequency-synchronized solution corresponds to a fixed point of (2),

$$\boldsymbol{\omega}-BW\mathbf{sin}(B^\top\boldsymbol{\theta}^\star)=\mathbf{0}_n. \tag{4}$$