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A 3D optimal control problem related to the urban heat islands

F.J. Fernández [a], L.J. Alvarez-Vázquez [b,*], A. Martínez [b], M.E. Vázquez-Méndez [c]

[a] *C.U.D. Escuela Naval Militar, 36920 Marín, Spain*
[b] *E.I. Telecomunicación, Universidade de Vigo, 36310 Vigo, Spain*
[c] *E.P.S. Universidade de Santiago de Compostela, 27002 Lugo, Spain*

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ABSTRACT

Within the framework of numerical simulation and optimal control of partial differential equations, in this work we deal with the mathematical modelling and control of the processes related to the urban heat island effect. In particular, we are interested in finding the optimal locations of green zones inside metropolitan areas in order to mitigate the consequences of this harmful phenomenon. So, we consider a three-dimensional climate model and formulate a constrained optimal control problem, that is extensively analyzed in the first part of the paper. Then, we propose a complete numerical algorithm for its resolution, interfacing the interior point algorithm IPOPT with the FreeFem++ software package. Finally, we present several numerical tests for a simple realistic case, where the advantages of our approach are shown.

1. Introduction

The phenomenon known as *urban heat island* (usually denoted as UHI) is characterized by higher temperatures in urban environments than in surrounding rural areas, mainly due to anthropogenic reasons. According to observations during past century, this temperature difference, which is primarily attributable to the urban built environment, ranges from 2 to 12 °C (most investigations refer to a 5 °C difference), and may pose particular risks to urban populations. The phenomenon is more remarkable during summer than in the other seasons of the year [25], and these temperature differences are larger at night than during the day, especially in the case of very weak winds.

Nowadays, UHI is considered as one of the major environmental challenges in this century as an undesired side effect of urbanization and industrialization of humankind, and many cities are adopting strategies to mitigate UHI effect, especially metropolises with large population and intensive economic activities (in many countries heat is the primary weather-related cause of death, and thus promotion of strategies for mitigating

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* Corresponding author.
E-mail addresses: fjavier.fernandez@cud.uvigo.es (F.J. Fernández), lino@dma.uvigo.es (L.J. Alvarez-Vázquez), aurea@dma.uvigo.es (A. Martínez), miguelernesto.vazquez@usc.es (M.E. Vázquez-Méndez).

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the UHI effect is a big concern for government agencies). The main reason for this night-time warming is that the shortwave radiation is still within concrete and asphalt (absorbed during the day), unlike rural areas, and this energy is slowly released during the night as longwave radiation, preventing a rapid cooling process. With a decreased amount of vegetation, cities often lose the shade and cooling effect of trees, the low albedo of their leaves, and the removal of carbon dioxide. Moreover, materials commonly used in urban areas for pavements and roofs, such as concrete and asphalt, have significantly different thermal bulk properties (including heat capacity and thermal conductivity) and surface radiative properties (albedo and emissivity) than the surrounding rural areas, leading to higher temperatures.

Mitigation of the UHI effect can be accomplished through the use of green roofs or of light-coloured surfaces in urban areas (which reflect more sunlight and absorb less heat), and also – as will be addressed in this paper – through the increasing of vegetation cover inside cities, mainly in the form of urban forests and parks, in order to maximize the multiple vegetation benefits in controlling the temperature rises. During last decades a vast research effort has been addressed and a wide range of literature is available for the subject, mainly from an engineering point of view (see, for instance, the review articles [28,4,27,20,18] and the numerous references therein). However, as far as we know, the mathematical approach to the problem has been much more poorly attended [17,1,21], and we can emphasize the recent work of the authors [12], where a bidimensional control problem has been formulated to mitigate in an optimal way the UHI effect by means of shadow trees zones.

This current study was carried out in order to introduce and develop a mathematical tool to deal with this environmental problem in a more realistic 3D framework. So, in section 2 we formulate a well-posed three-dimensional model for the UHI phenomenon, which is mathematically analyzed and numerically solved in sections 3 and 4, respectively. Sections 5 to 7 are devoted, respectively, to present the formulation of the optimal control problem, prove the existence of optimal solutions, and introduce a numerical algorithm for its computation. Finally, last sections of the paper are devoted to show several numerical results and conclusions for a realistic example.

2. Mathematical formulation of the problem

In this section we present the three-dimensional mathematical model that we will need to solve in order to address the optimal control problem. Basically, we consider four coupled systems of partial differential equations (mainly, those describing the velocity and the temperature of air, and the temperatures of soil and buildings) posed on three different domains whose union represents the whole domain under study (these three domains are, respectively, the air, the soil and the buildings).

2.1. The physical domain

We start this subsection introducing several notations related to the definition of the domain and its corresponding boundaries, whose graphical representation can be seen in Fig. 1.

We define the 2D domain $\Omega^{2D} = \{(x,y) \in \mathbb{R}^2 : \ 0 < x < a, \ 0 < y < b\}$ where we consider the positive functions H_S, $H_A : \overline{\Omega^{2D}} \longrightarrow \mathbb{R}^+$ representing, respectively, the height of the layers of soil and air, with $\mathbb{R}^+$ denoting the set of nonnegative real numbers, that is, $\mathbb{R}^+ = \{\alpha \in \mathbb{R} : \alpha \geq 0\}$. We also consider a subdomain $\Omega_B^{2D} \subset \Omega^{2D}$ (the place over where buildings are constructed), and a positive function $H_B : \overline{\Omega_B^{2D}} \longrightarrow \mathbb{R}^+$ representing the height of these buildings (obviously, we assume that $H_B(x,y) < H_A(x,y)$ for any $(x,y) \in \overline{\Omega_B^{2D}}$). Then, we define the following 3D domains:

- The domain occupied by soil:

$$\Omega_S^{3D} = \{\mathbf{x} = (x,y,z) \in \mathbb{R}^3 : \ (x,y) \in \Omega^{2D}, \ 0 < z < H_S(x,y)\}$$