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Large solutions for a nonhomogeneous quasilinear elliptic problem

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A R T I C L E I N F O

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A B S T R A C T

This paper focuses on a nonhomogeneous quasilinear elliptic boundary blow up problem

$$\triangle_p u = a(x)f(u) + h(x) \text{ in } \Omega, \quad u|_{\partial\Omega} = \infty,$$

where $p > 1$, Ω is a smooth bounded domain in $\mathbb{R}^n$, and $h \in C(\Omega)$ may be sign-changing in Ω and singular on $\partial\Omega$. We mainly analyze the influences caused by h on the existence of large solutions. Furthermore, it can be verified that any large solution is nonnegative under an additional assumption.

1. Introduction and main results

In this paper, we consider a nonhomogeneous quasilinear elliptic problem with blow up boundary value

$$\begin{cases} \triangle_p u = a(x)f(u) + h(x) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $p > 1$, Ω is a smooth bounded domain in $\mathbb{R}^n$ $(n \geq 1)$, the nonhomogeneous term h is continuous in Ω, the weight function $a \in C^{\beta}_{\text{loc}}(\Omega)$ $(\beta \in (0,1))$ is nonnegative in Ω and satisfies

$$\lim_{x\to\partial\Omega} \frac{a(x)}{d^{\sigma}(x)} = C_0 > 0 \quad (\sigma > -p), \tag{1.2}$$

the nonlinear term $f : \mathbb{R} \to \mathbb{R}$ satisfies

(f_1) f is continuous and nondecreasing in $\mathbb{R}$, $f(0) = 0$ and $\lim_{s\to-\infty} f(s) = -\infty$,

(f_2) $\lim_{s\to+\infty} \frac{f(ts)}{f(s)} = t^{p-1+\tau}$ $(\tau > 0)$ for any $t > 0$ and $f(s)s^{1-p}$ is nondecreasing in $(0, +\infty)$.

We say $u \in C(\Omega)$ is a large solution of (1.1) if u solves the equation in any compact subset of Ω in the weak sense and satisfies the boundary condition in the sense that $u(x) \to +\infty$ as $d(x) \to 0$, where $d(x) = \text{dist}(x, \partial\Omega)$.

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Boundary blow up problems arise in various subjects such as electric potential in some bodies, subsonic motion of gas and Riemannian geometry. There are an enormous of works focusing on these problems, and the applications of them have been investigated in [1,2], we also refer to [3] for the comprehensive description of large solutions.

A simple form of (1.1) follows from $h = 0$ in Ω and $p = 2$, that is,

$$\begin{cases} \triangle u = a(x)f(u) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

The case that the weight function $a(x) \equiv 1$ in Ω has been widely analyzed. Let us simply recall the main results. For $n = 2$ and $f(u) = e^u$, Bieberbach [4] firstly established a solution $u \in C^2(\Omega)$ to (1.3) such that $|u(x) - \ln(d(x))^{-2}|$ is bounded in Ω. A similar result was obtained by Rademacher [5] for $n = 3$. And for $f(u) = u^{p_0}$, $p_0 = \frac{n+2}{n-2}$ $(n > 2)$, Loewner and Nirenberg [6] proved that (1.3) has only one large solution satisfying

$$\lim_{d(x)\to 0} u(x)(d(x))^{\frac{n-2}{2}} = \left(\frac{n(n-2)}{4}\right)^{\frac{n-2}{4}}.$$

For the general nonlinear terms f, Keller [7] and Osserman [8] gave a sufficient and necessary condition to the existence of large solutions, which is

(f_3) $\int_t^\infty \frac{ds}{\sqrt{2F(s)}} < \infty$, $\forall t > 0$, where $F(s) = \int_0^s f(z)dz$.

Bandle and Marcus [9] investigated the boundary behavior and the uniqueness of large solutions by analyzing the corresponding ordinary differential equation and using the maximum principle. Moreover, a sign-changing large solution was obtained in [10].

Later, there are many works focusing (1.3) on the case that the weight function is not a constant. Under the assumption that $a(x)$ is continuous and strictly positive on $\overline{\Omega}$, Lazer and Mckenna [11] continued the works of Bieberbach and Rademacher on a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ with the uniform external sphere condition. One can also see [12–15] and the references therein for the case that either $a(x)$ is positive in Ω and can be vanishing (or even singular) on $\partial\Omega$ or $a(x)$ satisfies

(a_1) $a \in C(\overline{\Omega})$ is nonnegative nontrivial in Ω, and if there is some $x_0 \in \Omega$ with $a(x_0) = 0$, then there exists $\Omega_0 \subset \Omega$ such that $x_0 \in \Omega_0$ and $a(x) > 0$, $\forall x \in \partial\Omega_0$.

For example, Lair [16] established a sufficient and necessary condition (f_3) for the existence of large solutions to (1.3) under the conditions (a_1) and (f_1). Recently, by applying the Karamata regular variation theory and constructing the comparison functions, Cîrstea [12] and Zhang [14] considered the uniqueness and boundary behavior of the large solutions.

There are also many papers extending the above results to the case that $p > 1$, namely, the homogeneous problem of (1.1). It was firstly studied by Diaz and Letelier [17] for $a \equiv 1$ in Ω and $f(u) = u^\rho$ with $\rho > p - 1$. The boundary behavior was investigated in [18] in the case that the weight function $a \equiv 1$ in Ω. Mohammed [19,20] considered the existence and boundary behavior of large solutions under assumptions that the weight function a is nonnegative and the nonlinear term f is regularly varying at infinity. Moreover, Repovš [21] refined the blow up rate of nonnegative large solutions by establishing the second order expansion near the boundary of Ω, which involves both the distance function $d(x)$ and the mean curvature of $\partial\Omega$. For other and further works, see [15,19,20,22–25] and the references therein.

Let us return to the problem (1.1). A complete analysis of the problem for general $p > 1$ and general continuous functions h has not been carried out except for the case $p = 2$ in [13,26,27]. Véron [27] considered the case that $p = 2$ and h is nonnegative and small when compared with $d(x)^{-\frac{2p}{p-1}}$ near $\partial\Omega$. In [26],