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Decision Support

Investigation on irreducible cost vectors in minimum cost arborescence problems

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ABSTRACT

We study cost allocation rules in minimum cost arborescence problems, where agents need to build a network to a source in order to obtain some resource. Provided a vector of costs of edges (agent/source pairs), the agents cooperate to construct a minimum cost arborescence rooted at the source in order to reduce the total building cost. Minimum cost arborescence problems are extensions of well-studied minimum cost spanning tree problems to deal with asymmetric edge costs. Regarding cost allocation rules in minimum cost arborescence problems, Dutta and Mishra (2012) extended the folk rule, which is one of the most important rules in minimum cost spanning tree problems, based on the problem with the vector of the most reduced costs, called irreducible form. In minimum cost spanning tree problems, several axiomatic characterizations of the folk rule have been proposed. However, it is difficult to extend them in minimum cost arborescence problems. One of the reasons is that strong and reasonable axioms in minimum cost spanning tree problems, which imply irreducible-form dependence of cost allocation rules, are not satisfied by the folk rule in minimum cost arborescence problems. Hence, we search for other axioms which imply irreducible-form dependence. For this purpose, we investigate irreducible cost vectors in minimum cost arborescence problems, and characterize the irreducible form.

1. Introduction

Cost allocation problems arise when agents jointly work to reduce the total cost of a project. There are many studies to deal with them by cooperative game theory or axiomatic approach. Among them, the studies for minimum cost spanning tree problems (mcstp, for short) are prominent. An mcstp is a situation where agents need to be connected to a source in order to obtain some resource. The connection of each agent is allowed to involve other agents. This situation is modeled by a graph where the nodes correspond to the agents and source, and a cost vector on the edges. The agents cooperate to minimize the total cost of a network which connects all the agents to the source. Such a network should be a minimum cost spanning tree (mcst, for short).

In the literature of mcstp, several researchers have proposed and characterized cost allocation rules. When mcstp are associated with transferable utility (TU) games, we can discuss cost allocation rules by solution concepts of cooperative game theory. The following cost game is usually considered: the value of each coalition of agents is the cost of mcst on the coalition independently constructed from the other agents.

The folk rule is one of the most important cost allocation rules, which is defined from several viewpoints (Bergantiños & Vidal-Puga, 2007a; Bogomolnaia & Moulin, 2010; Branzei, Moretti, Norde, & Tijs, 2004; Feltkamp, Tijs, & Muto, 1994). Among the definitions, Bergantiños and Vidal-Puga (2007a) defined the folk rule by the Shapley value of the cost game associated with the irreducible form of a given mcstp. Here, the irreducible form is an mcstp with the most reduced cost vector preserving the cost of mcst of the original problem. Additionally, if any costs of edges cannot be reduced without reducing the cost of mcst, the mcstp and its cost vector are called irreducible. The folk rule satisfies many desirable properties (Bergantiños & Vidal-Puga, 2007a; Tijs, Branzei, Moretti, & Norde, 2006). Moreover, the folk rule and its generalizations are characterized by several axiomatic systems (Bergantiños & Kar, 2010; Bergantiños & Vidal-Puga, 2007a; 2015; Branzei et al., 2004; Feltkamp et al., 1994). By the definition, the folk rule depends only on the irreducible form. This property of cost allocation rules is called reductionism.

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Regarding the folk rule for mcstp, Bergantiños and Vidal-Puga (2007b) proposed another TU game, called the optimistic cost game, where the value of each coalition is the cost of an mcst on the coalition supposing that the other agents are already connected to the source and pay the costs of the connections. The optimistic cost game is dual of the cost game of the irreducible form (Bergantiños & Vidal-Puga, 2007b). Hence, by self-duality of the Shapley value, the folk rule can be expressed by the Shapley value of the optimistic cost game.

In mcstp, the costs of the edges are required to be symmetric. That is, letting i and j be an agent/source pair, and c_{ij} and c_{ij} be the costs of edges from i to j and from j to i, respectively, we suppose $c_{ij} = c_{ji}$. However, there are some situations where this assumption is violated. For example, consider a project to construct an irrigation system which draws water from a dam to several villages (Dutta & Mishra, 2012). When a pair of villages i and j is situated at different altitudes, the cost of the canal from i to j is different to that from j to i. To take asymmetric edge costs into account, we should consider minimum cost arborescences (mca) rooted at the source instead of mcst. Such problems are called minimum cost arborescence problems (mcap). The difference between mcap and mcstp is that mcap allow asymmetric cost vectors, that is, $c_{ij} = c_{ji}$ is not required in mcap. This paper addresses cost allocation rules in mcap.

Recently, Dutta and Mishra (2012) studied cost allocation rules in mcap. They proposed the folk rule for mcap as the Shapley value of the cost game of a special irreducible form. In mcap, there can be more than one irreducible form for an mcap. We call the irreducible form used in the folk rule Dutta–Mishra (DM) irreducible form. They characterized the folk rule, although in a partial domain.

As mentioned above, several axiomatic characterizations have been proposed in mcstp. However, it is difficult to extend them in mcap. One of the reasons is that strong and reasonable axioms implying the reductionism in mcstp (e.g. strong cost monotonicity or independence of irrelevant trees (Bergantiños & Vidal-Puga, 2007a)), are not satisfied by the folk rule in mcap. If cost allocation rules necessarily have the reductionism, construction of axiomatic systems becomes easier, since we only deal with cost allocation rules on the domain of irreducible mcap. Hence, we search for other axioms which imply the reductionism with respect to the DM irreducible form. For this purpose, we investigate irreducible cost vectors in mcap, and reveal what is the DM irreducible form.

First, we discuss a relationship between irreducibleness and laminarity. Let N be the set of the agents. We introduce a laminar vector $(d_S)_{\emptyset \neq S \subseteq N}$ which is a nonnegative vector on all the nonempty subsets of the agent set N with an additional property. The property is that the family of all the subsets S taking positive values $d_S > 0$ forms a laminar or nested structure. Additionally, we introduce an optimization problem to solve mcap. Then, we prove that there is a unique optimal laminar solution for the dual problem of the optimization problem with respect to each irreducible cost vector. Combining it with the fact that the cost vector derived from each laminar vector is irreducible, we can establish a one-to-one correspondence between the set of irreducible cost vectors and the set of laminar vectors.

Moreover, we prove that the vectors of Harsanyi dividends and dual Harsanyi dividends, which are important concepts in cooperative game theory, of the cost game and the optimistic cost game, respectively, derived from an irreducible cost vector coincide with the laminar vector corresponding to the irreducible cost vector. It implies that there are also one-to-one correspondences between the set of irreducible cost vectors and the sets of irreducible cost games/optimistic irreducible cost games.

Using the above results, we derive the main result of this paper as follows.

Theorem. Let c be a cost vector. The DM irreducible form of c is the unique irreducible form of c preserving the optimistic cost game derived from c.

This theorem implies that a cost allocation rule depends only on the DM irreducible form if and only if it depends only on the optimistic cost game. We also discuss other consequences of our findings, which are related to the folk rule. Among the consequences, we remark that polynomial time complexity of the folk rule which is not explicitly mentioned in Dutta and Mishra (2012), and that the folk rule is the Shapley value of the optimistic cost game which is the parallel result to Bergantiños and Vidal-Puga (2007b) for mcstp.

Although the study of this paper does not archive axiomatic characterizations of the folk rule, the results will provide possibility of more constructive studies of cost allocation rules in mcap. For example, if we accept "optimistic-cost-game-dependence" axiom that the cost allocations for different mcap inducing the same optimistic game should be identical, we may construct axiomatic characterizations of the folk rule for mcap in the similar manner as characterizations in mcstp (Bergantiños & Vidal-Puga, 2007a; Trudeau, 2013), where the reductionism plays an important role. Furthermore, using the optimistic-cost-game-dependence axiom, we may characterize generalized folk rules for mcap which are defined by weighted Shapley values of the optimistic cost game. Such generalizations of the folk rule for mcstp were proposed in Bergantiños and Lorenzo-Freire (2008b) and characterized in Bergantiños and Lorenzo-Freire (2008a). We remark that, using the results of this paper, we can show that the generalized folk rules are obtained in polynomial time. Additionally, Gómez-Rúa and Vidal-Puga (2016) characterized the weighted Shapley value of the optimistic cost game applied to generalized mcstp where weights represent the numbers of agents assigned to the nodes of a network. We may propose characterizations of the rule in mcap-counterparts of those situations using the axiom.

This paper is organized as follows. In Section 2, introducing graphs and cooperative games, we define mcap and the cost games associated with mcap. We also provide basic results for mcap in the viewpoint of optimization. Moreover, we define irreducible mcap and the folk rule. In Section 3, we establish a bijection between the set of irreducible cost vectors and the set of laminar vectors. In Section 4, we provide explicit representations of the cost game and the optimistic cost game derived from an irreducible cost vector using the laminar vector determined by the bijection of Section 3. The main theorem is proved in Section 5 using the results of Sections 3 and 4. In Section 5, we also show some properties of the folk rule as a consequence of the results of Sections 3 and 4. Finally, we provide concluding remarks in Section 6.

2. Minimum cost arborescence problems

In this section, we first introduce graphs, set systems and related concepts. And then, we define minimum cost arborescence problems. After that we introduce cooperative games and define cost games associated with minimum cost arborescence problems. Finally, we define irreducible cost vectors and the folk rule.

2.1. Graphs and set systems

We describe definitions used in graphs. Let V be a finite set, whose elements are called nodes or vertices. For every pair i, $j \in V$, ij is the (directed) edge from i to j. We define the set of all edges $V^{\bar{2}} = \{ij \mid i, j \in V, i \neq j\}$. For $E \subseteq V^{\bar{2}}$, a pair $G = (V, E)$ is called a graph. For a graph G, $V(G)$ and $E(G)$ denote the node set of G and the edge set of G, respectively. A subgraph of $G = (V, E)$ is a graph