Physica A 391 (2012) 3971–3977

Contents lists available at SciVerse ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Time-changed geometric fractional Brownian motion and option pricing with transaction costs

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ARTICLE INFO

Article history:
Received 21 October 2011
Received in revised form 20 February 2012
Available online 26 March 2012

Keywords:
Option pricing
Transaction costs
Delta-hedging
Time-changed process
Inverse α-stable subordinator

ABSTRACT

This paper deals with the problem of discrete time option pricing by a fractional subdiffusive Black–Scholes model. The price of the underlying stock follows a time-changed geometric fractional Brownian motion. By a mean self-financing delta-hedging argument, the pricing formula for the European call option in discrete time setting is obtained.

1. Introduction

Since it appeared in 1973, the Black–Scholes (BS) model [1] has become the most popular method for option pricing. The BS model consists of two assets: a riskless bond with constant interest rate and a stock whose price follows a geometric Brownian motion (GBM)

$$X(t) = X_0 \exp\{\mu t + \sigma B(t)\}, \quad X_0 > 0, \tag{1}$$

with constant rate of return μ and volatility σ, where $B(t)$ is the standard Brownian motion. The process X can be equivalently defined in the form of the stochastic differential equation

$$\mathrm{d}X(t) = \left(\mu + \frac{\sigma^2}{2}\right) X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}B(t), \quad X(0) = X_0 > 0. \tag{2}$$

Nowadays, the BS model is still classical and most popular model of the market. However, empirical research shows that it cannot capture many of the characteristic features of prices, such as: long-range correlations, heavy-tailed and skewed marginal distributions, lack of scale invariance, periods of constant values, etc. Therefore, improvements of the BS model itself did not stand still either.

In the paper [2], Magdziarz applied the subdiffusive mechanism of trapping events to describe properly financial data exhibiting periods of constant values and introduced the subdiffusive geometric Brownian motion (SGBM)

$$X_\alpha(t) = X(T_\alpha(t)) \tag{3}$$

as the model of asset prices exhibiting subdiffusive dynamics. Here the parent process $X(\tau)$ is the GBM defined in (1) or (2), $T_\alpha(t)$ is the inverse α-stable subordinator with the parameter $\alpha \in (0, 1)$. Moreover, $T_\alpha(t)$ is assumed to be independent

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doi:10.1016/j.physa.2012.03.020

of the Brownian motion $B(t)$. To see more models which describe such characteristic behavior, one can refer the relevant papers [3–8]. Magdziarz showed that the considered model is arbitrage-free but incomplete, and obtained the corresponding subdiffusive BS formula for the fair prices of European options.

The fractional BS model is another generalization of the BS model, which displays the long-range dependence observed in empirical data. This model is based on replacing in (1) the classic Brownian motion by the fractional Brownian motion (FBM). That is

$$\widetilde{X}(t) = \widetilde{X}_0 \exp\{\mu t + \sigma B_H(t)\}, \quad \widetilde{X}_0 > 0, \tag{4}$$

here $B_H(t)$ is a standard FBM with Hurst exponent $H \in (\frac{1}{2}, 1)$. It has been shown that the fractional BS model admits arbitrage in a complete and frictionless market [9,10]. Wang [11] resolved this contradiction through giving up the arbitrage argument and examining option replication in the presence of proportional transaction costs in a discrete time setting [12].

In this paper, inspired by the work of Wang [11,13–15] and Magdziarz [2], we consider a time-changed fractional BS model, whose price of underlying stock is

$$S_t = \widetilde{X}(T_\alpha(t)) = S_0 \exp\{\mu T_\alpha(t) + \sigma B_H(T_\alpha(t))\}, \quad S_0 = \widetilde{X}(0) > 0. \tag{5}$$

The following figures (see Fig. 1) typically describe the differences and relations between trajectories of the stock prices in the fractional BS model and the time-changed fractional BS model.

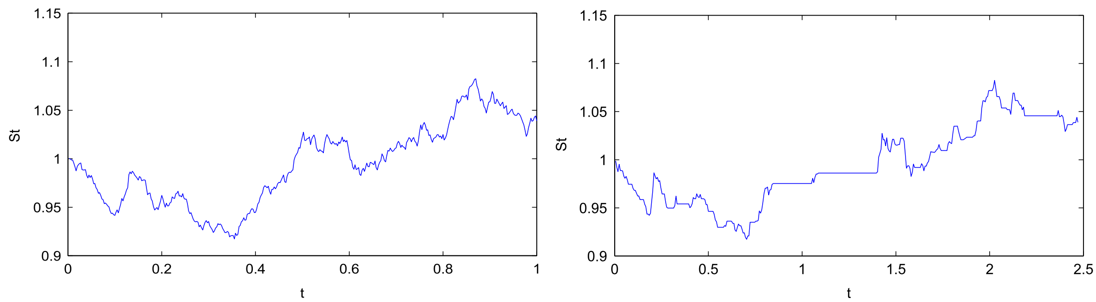

Fig. 1. Comparison of the trajectories of the stock prices in the fractional BS model (left) and the time-changed fractional BS model (right) for $\alpha = 0.8$, $H = 0.55$, $\mu = 0.05$, $S_0 = 1$ and $\sigma = 0.1$.

The paper is organized as follows. In Section 2, some properties of the time-changed fractional BS model are introduced. In Section 3, we investigate the option pricing under transaction costs in a discrete time setting and obtain the option formula for the European call option when the timestep Δt is given. Section 4 is the conclusion.

2. Time-changed geometric FBM

The time-change process considered is the inverse of an α-stable subordinator [16,17] $U_\alpha(\tau)$. A Lévy process $\{U(\tau)\}_{\tau\geq 0}$ with nonnegative increments is called a subordinator and its inverse is the first-passage time process defined as $T(t) = \inf\{\tau > 0 : U(\tau) > t\}$. Specially, for $\alpha \in (0, 1)$, $\{U_\alpha(\tau)\}_{\tau\geq 0}$ with Laplace transform: $\mathbb{E}\left(e^{-uU_\alpha(\tau)}\right) = e^{-\tau u^\alpha}$ is called an α-stable subordinator and its inverse $T_\alpha(t)$ is called an inverse α-stable subordinator. $U_\alpha(t)$ is $\frac{1}{\alpha}$-self-similar and $T_\alpha(t)$ is α-self-similar, that is, for every $c > 0$, $U_\alpha(ct) \overset{d}{=} c^{\frac{1}{\alpha}} U_\alpha(t)$, $T_\alpha(ct) \overset{d}{=} c^\alpha T_\alpha(t)$, where $\overset{d}{=}$ denotes "is identical in law to" or "has the same finite-dimensional distributions as". The trajectories of $T_\alpha(t)$ are continuous a.s., non-decreasing and singular with respect to the Lebesgue measure. The moments of $T_\alpha(1)$ satisfies $\mathbb{E}(T_\alpha^n(1)) = \frac{n!}{\Gamma(n\alpha+1)}$ for any $n \in \mathbb{N}$. To see more about the α-stable subordinator and its inverse, one can refer [18–21].

Consider the subordinated process $Z_{\alpha,H}(t) = B_H(T_\alpha(t))$, here the parent process $B_H(\tau)$ is a standard FBM and $T_\alpha(t)$ is assumed to be independent of $B_H(\tau)$. We call $Z_{\alpha,H}(t)$ a subdiffusion process. Specially, when $H = \frac{1}{2}$, it is a subdiffusion process mentioned in Refs. [22,23]. The process defined by (4) is a fractional geometric FBM, if we replace the time t by the time-change process $T_\alpha(t)$, we get a time-changed geometric FBM defined in (5).

For $\beta > 0$, a random function $X(x)$ is said to be $o(x^\beta)$ if $\lim_{x\downarrow 0} \frac{\mathbb{E}(|X(x)|^n)}{x^{n\beta}} = 0$ for every $n \in \mathbb{N}$. The following properties are obvious: for $0 < \beta, \beta_1, \beta_2 < +\infty$ and $n \in \mathbb{N}$

$$o(x^{\beta_1}) \cdot o(x^{\beta_2}) = o(x^{\beta_1+\beta_2}), \quad \text{in particular } \left(o(x^\beta)\right)^n = o(x^{n\beta}), \tag{6}$$

$$o(x^{\beta_1}) + o(x^{\beta_2}) = o(x^{\min\{\beta_1,\beta_2\}}). \tag{7}$$

The next lemma is about the increments of the inverse α-stable subordinator and the fractional subdiffusion process.