Journal of Econometrics 167 (2012) 211–223

Contents lists available at SciVerse ScienceDirect

Journal of Econometrics

journal homepage: www.elsevier.com/locate/jeconom

The conditional autoregressive Wishart model for multivariate stock market volatility

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ARTICLE INFO

Article history:
Received 9 June 2010
Received in revised form
3 November 2011
Accepted 10 November 2011
Available online 18 November 2011

JEL classification:
C32
C58
G17

Keywords:
Component volatility models
Covariance matrix
Mixed data sampling
Observation-driven models
Realized volatility

ABSTRACT

We propose a Conditional Autoregressive Wishart (CAW) model for the analysis of realized covariance matrices of asset returns. Our model assumes an autoregressive moving average structure for the scale matrix of the Wishart distribution. It accounts for positive definiteness of covariance matrices without imposing parametric restrictions, and can be estimated by Maximum Likelihood. We also propose extensions of the CAW model obtained by including a Mixed Data Sampling (MIDAS) component and Heterogeneous Autoregressive (HAR) dynamics for long-run fluctuations. The CAW models are applied to realized variances and covariances for five New York Stock Exchange stocks.

1. Introduction

Multivariate modeling and forecasting the variances and covariances of asset returns play a prominent role in many practical situations, ranging from portfolio allocation and asset pricing to risk assessment. In practice, the covariance matrix of asset returns is not directly observable and most existing models treat it either as measurable given past observations, such as multivariate GARCH models introduced by Bollerslev et al. (1988), or as an inherently latent quantity, such as multivariate stochastic volatility (SV) models introduced by Harvey et al. (1994). Excellent overviews on multivariate GARCH and SV models can be found in Bauwens et al. (2006) and Asai et al. (2006), respectively. An alternative approach of covariance estimation and modeling, which has attracted substantial interest in recent years, uses high-frequency return data to construct the realized variances and covariances as precise estimates for the variances and covariances of low-frequency returns (see e.g., Andersen et al., 2003; Barndorff-Nielsen and Shephard, 2004). As such, the observed realized variances and covariances can be modeled directly as advocated, for example, by Andersen et al. (2003). Multivariate models for the realized covariance matrix should satisfy two important requirements, namely that the predicted covariance matrices remain positive definite, and, second, that the specification is parsimoniously parameterized yet empirically realistic with the ability to account for the strong serial dependence typically observed for realized variances and covariances.

Pioneering multivariate approaches to model the dynamics in the realized covariance matrix are found in Gourieroux et al. (2009), Jin and Maheu (2009), Chiriac and Voev (2011), and Bauer and Vorkink (2011). The specification proposed by Gourieroux et al. (2009) extends the Wishart distribution of the sample covariance for i.i.d. multivariate Gaussian random variables by allowing the multivariate Gaussian random variables to be serially correlated. Under the resulting Wishart Autoregressive (WAR) process the realized covariance has a transition distribution which is non-central Wishart with a non-centrality parameter depending on lagged covariances and a fixed scale matrix. As such the WAR model naturally accommodates the positive definiteness of predicted covariance matrices without any parametric restrictions. The approach followed by Jin and Maheu (2009) also relies on a Wishart transition distribution, but assumes a central rather than a non-central Wishart distribution, and decomposes its scale matrix into multiplicative components, which are driven by

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doi:10.1016/j.jeconom.2011.11.004

sample averages of lagged realized covariance matrices. In order to account for the positive definiteness, the approaches of Chiriac and Voev (2011) and Bauer and Vorkink (2011) use appropriate transformations of the covariance matrix. The former approach is based upon a Cholesky decomposition of the covariance matrix and assumes fractionally integrated processes for the individual elements of the Cholesky factor. The latter approach transforms the covariance matrix by using the matrix logarithm function and specifies the individual elements of the transformation as functions of latent factors driven by lagged volatilities and lagged returns.

In the present paper, we adopt a novel conditional autoregressive Wishart (CAW) approach and propose a new flexible dynamic model for the realized covariance matrix of asset returns. Its baseline specification assumes a simple autoregressive moving average structure for the scale matrix of the Wishart distribution allowing to account for complex serial dependences in the variances and covariances. In particular, under our model the predicted covariance matrix depends on lagged covariance matrices as well as on their lagged predictions. As such it presents a dynamic generalization of the models proposed by Gourieroux et al. (2009) and Jin and Maheu (2009), where the predicted covariance matrix is specified as a function of lagged covariances only. Our model also accounts for symmetry and positive definiteness of the predicted covariance matrices without imposing parametric restrictions and can easily be estimated by Maximum Likelihood (ML). In addition, it allows us to derive in a straightforward manner conditions for stationarity and other important time series properties. A further advantage of our approach is that its baseline specification can easily be generalized. The extensions of the baseline CAW model we explore, are specifically designed to capture the long-run fluctuations in the variances and covariances. For this purpose, we combine the CAW specification with the mixed data sampling (MIDAS) approach of Ghysels et al. (2005, 2006) and, alternatively, with a heterogeneous autoregressive (HAR) component as used by Corsi (2009) and Bonato et al. (2009).

The rest of the paper is organized as follows. In Section 2 we introduce the baseline CAW model and discuss its stochastic properties. Extensions of the baseline model are proposed in Section 3. The empirical application to NYSE data is presented in Section 4. Section 5 concludes. The proofs are provided in the Appendix.

2. Conditional Autoregressive Wishart (CAW) model

2.1. CAW(p, q) model

Consider the stochastic, symmetric positive definite matrix $R_t = (r_{ij,t})$ of realized covariances with dimension $n \times n$ recorded at time t ($t = 1, \ldots, T$). The matrix R_t given the past history $\mathcal{F}_{t-1} = \{R_{t-1}, R_{t-2}, \ldots\}$ is assumed to follow a central Wishart distribution

$$R_t|\mathcal{F}_{t-1} \sim \mathcal{W}_n(\nu, S_t/\nu), \tag{1}$$

where $\nu > n$ is the scalar degree of freedom, and S_t/ν is the $n \times n$ symmetric, positive definite scale matrix with $S_t = (s_{ij,t})$, such that the conditional mean and covariances are (see Muirhead, 1982)

$$\begin{aligned} E(R_t|\mathcal{F}_{t-1}) &= S_t, \\ \mathrm{Cov}(r_{ij,t}, r_{lm,t}|\mathcal{F}_{t-1}) &= \frac{1}{\nu}(s_{il,t} \cdot s_{jm,t} + s_{im,t} \cdot s_{jl,t}) \end{aligned} \tag{2}$$

for $i, j, l, m = 1, \ldots, n$. The density function for $R_t|\mathcal{F}_{t-1}$ has the form

$$f(R_t|\mathcal{F}_{t-1}) = \frac{|S_t/\nu|^{-\nu/2}|R_t|^{(\nu-n-1)/2}}{2^{\nu n/2}\pi^{n(n-1)/4}\prod_{i=1}^{n}\Gamma([\nu+1-i]/2)} \times \exp\left\{-\frac{1}{2}\mathrm{tr}(\nu S_t^{-1}R_t)\right\}, \tag{3}$$

where $\Gamma(\cdot)$ denotes the Gamma function. In order to account for serial- and cross-correlation across the elements in R_t we assume that the matrix-variate process S_t follows the linear recursion of order (p, q)

$$S_t = CC' + \sum_{i=1}^{p} B_i S_{t-i} B_i' + \sum_{j=1}^{q} A_j R_{t-j} A_j', \tag{4}$$

where C is a $n \times n$ lower-triangular matrix and A_j, B_i are $n \times n$ parameter matrices. This recursion of order (p, q) resembles the BEKK-GARCH(p, q) specification of Engle and Kroner (1995) for the conditional covariance in models for multivariate returns, and has the appealing property to guarantee the symmetry and positive-definiteness of the conditional mean S_t essentially without imposing parametric restrictions on (C, A_j, B_i) as long as the initial matrices $S_0, S_{-1}, \ldots, S_{-p+1}$ are symmetric and positive definite. In fact, S_t is positive definite if the null spaces of $C', A_1', \ldots, A_q'$ and $B_1', \ldots, B_p'$ all intersect only at the origin (see Engle and Kroner, 1995, Proposition 2.5).

In the CAW(p, q) model defined by Eqs. (1) and (4), the degree of freedom of the Wishart distribution ν is associated with the overall variability of the covariances in R_t, which is decreasing in ν. The joint dynamic behavior of the covariances is directed by the parameters in the A_j and B_i matrices. However, note that those parameters do not have direct interpretations in terms of their impact on the conditional mean S_t, as they enter the model in a quadratic form. In the bivariate case with $n = 2$ and ($p = 0$, $q = 1$), for example, the elements in S_t obtain as

$$\begin{aligned} s_{11,t} &= c_{11}^2 + a_{11}^2 r_{11,t-1} + 2a_{11}a_{12}r_{12,t-1} + a_{12}^2 r_{22,t-1}, \\ s_{12,t} &= c_{11}c_{21} + a_{11}a_{21}r_{11,t-1} + (a_{11}a_{22} + a_{12}a_{21}) \\ &\quad \times r_{12,t-1} + a_{12}a_{22}r_{22,t-1}, \\ s_{22,t} &= c_{22}^2 + c_{21}^2 + a_{22}^2 r_{22,t-1} + 2a_{22}a_{21}r_{12,t-1} + a_{21}^2 r_{11,t-1}. \end{aligned} \tag{5}$$

This representation additionally shows that the 9 autoregressive coefficients representing the marginal effects of the lagged $r_{ij,t}$s are parameterized using 4 parameters only. Hence, the CAW(p, q) model imposes over-identifying restrictions on the autoregressive coefficients.

Moreover, note that under the CAW model not only the conditional means of the elements in R_t but also their conditional variances and covariances given by Eq. (2) are driven by lagged observations via the recursion (4), which generates nonlinear serial dependence in the time series behavior of R_t.

The CAW(p, q) specification can be interpreted as a state-space model with S_t as a state variable measured by the observable matrix R_t and with measurement density given by Eq. (3). The corresponding measurement equation obtains as (see Muirhead, 1982, p. 95, Theorem 3.2.11)

$$R_t = \frac{1}{\nu} S_t^{1/2} U_t (S_t^{1/2})', \quad U_t \sim \mathcal{W}_n(\nu, I_n), \tag{6}$$

where $S_t^{1/2}$ denotes the lower-triangular Cholesky factor of S_t such that $S_t = S_t^{1/2}(S_t^{1/2})'$ and U_t represents the measurement error following a standardized Wishart distribution with ν degree of freedom and a scale matrix given by the identity matrix I_n. This allows us to interpret S_t as the 'true' integrated covariance for a broad class of multivariate continuous-time stochastic volatility