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Finite change comparative statics for risk-coherent inventories

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ABSTRACT

This work introduces a comprehensive approach to the sensitivity analysis (SA) of risk-coherent inventory models. We address the issues posed by (i) the piecewise-defined nature of risk-coherent objective functions and (ii) by the need of multiple model evaluations. The solutions of these issues are found by introducing the extended finite change sensitivity indices (FCSI's). We obtain properties and invariance conditions for the sensitivity of risk-coherent optimization problems. An inventory management case study involving risk-neutral and conditional value-at-risk (CVaR) objective function illustrates our methodology. Three SA settings are formulated to obtain managerial insights. Numerical findings show that risk-neutral decision-makers are more exposed to variations in exogenous variables than CVaR decision-makers.

1. Introduction

Recent works have demonstrated the use of coherent measures as a novel and effective way to manage risk in inventory problems Ahmed et al. (2007), Gotoh and Takano (2007), Borgonovo and Peccati (2009a). The convexity of the objective functions insures feasibility in a broad variety of applications. However, an explicit expression of the solution is generally not available. This prevents a direct interpretation of model results and a straightforward derivation of managerial insights.

The need to explain "*what it was about the inputs that made the outputs come out as they did* Little (1970); p. B469" is underlined in Little's seminal paper on the creation and utilization of decision-support models for managers. Eschenbach (1992) underlines the need of identifying the "*most critical factors*" on which to focus "*managerial attention during implementation* (Eschenbach, 1992; pp. 40–41)." Works as Rabitz and Alis (1999), Wallace (2000), Saltelli et al. (2000), Saltelli and Tarantola (2002), Saltelli et al. (2004) have established the awareness that these questions are answered only by a systematic application of sensitivity analysis (SA).

Wallace (2000) and Higle and Wallace (2003) address the use of SA in examining management science model output. They underline the key-issue of establishing consistency between the managerial questions and the SA method selected for the analysis. In linear programming, Jansen et al. (1997), Koltai and Terlaky (2000), Koltay and Tatay (2008) discuss the differences in the mathematical and managerial interpretation of SA results. Saltelli et al. (2008) (p. 24) recognize that "*a poor definition of the objectives of a sensitivity analysis can lead to confused or inconclusive results.*" The works by Saltelli and Tarantola (2002), Saltelli et al. (2004) and Saltelli et al. (2008) demonstrate that these issues are solved by SA settings. A setting is "*a way of framing the sensitivity quest in such a way that the answer can be confidently entrusted to a well-identified sensitivity measure* Saltelli et al. (2008), p. 24."

Purpose of this work is to establish a comprehensive and consistent approach to the SA of risk-coherent inventory problems. To achieve this goal, we proceed as follows. We first address the specific (1) technical and (2) result communication issues. Technical issues are posed by the piecewise-defined character of risk-coherent objective functions Borgonovo and Peccati (2009b). This non-smoothness makes comparative statics and differential approaches not applicable. We show that the integral function decomposition at the basis of the finite change sensitivity indices (FCSI) provides the required generality and solves the technical issues. Result communication issues are posed by the multi-item nature of the problem and, more in general, by the presence of multiple outputs of interests to the decision-maker. We introduce two alternative ways for dealing with result communication. The utilization of the norm of the optimal policy and the technique of the Savage Score correlation coefficients Iman and Conover (1987). We highlight advantages and drawbacks of each approach.

The second step is to enrich information further by enabling a deeper exploration of the exogenous variable space. In SA practice, decision-makers assess a set of efficient scenarios Tietje (2005). The model is tested at each scenario. Since, in previous inventory management works one [in perturbation approaches Bogataj and Cibej (1994), in comparative statics Borgonovo (2008) or two points (Borgonovo, 2010) were explored, we need to

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formalize the application of FCSI's in the presence of multiple scenarios. We show that this is achieved by applying the finite-change decomposition at each model jump. As a result, plentiful information is obtained on the behavior of the decision criteria and on the determinants of the problem. We synthesize this information in sensitivity measures called extended FCSI's. By the extended FCSI's one obtains insights on both the magnitude and direction of impact and on the importance of the exogenous variables. Flexibility in assessing the effect of individual variables and groups is offered by the approach.

The third step is to derive general properties of extended FCSI's in risk-coherent problems. We show that, if the loss function of the system at hand (not necessarily an inventory system) is separable in a group of exogenous variables, then: (i) the optimal risk-coherent policy is insensitive on that group; (ii) the value of the risk-measure at the optimum is sensitive and responds additively to changes in the parameters of the group.

We then discuss the SA settings that allow one to interpret numerical results and obtaining managerial insights consistence with Eschenbach's and Little's questions.

We apply the proposed methodology to a stochastic inventory problem with risk-neutral and conditional value at risk (CVaR) objective functions. Numerical results confirm the theoretical expectations on the behavior of the sensitivity measures. We discuss managerial insights in the light of the SA settings. Comparison of the numerical findings for risk-neutral and CVaR decision-makers show that both the CVaR optimal policy and value-at-the-optimum are less sensitive to exogenous variable changes than the corresponding risk-neutral optimal policy and expected loss.

The remainder of this work is organized as follows. Section 2 discusses technical aspects and the choice of the sensitivity measures. Section 3 formalizes the notion of extended FCSI's. Section 4 proves relevant properties of the sensitivity of risk-coherent problems under separability conditions of the loss function. Section 5 discusses the SA settings for gaining managerial insights. Section 6 presents the case study and illustrated numerical findings. Conclusions are offered in Section 7.

2. Comparative statics in risk-coherent problems: issues and solutions

In this section, we address technical aspects associated with the piecewise-definite nature of risk-coherent objective functions Borgonovo and Peccati (2009b).

We start with a deterministic inventory system as in Borgonovo (2008), to examine the conditions under which comparative statics is applicable. Let $\mathbf{y} \in Y \subseteq \mathbb{R}^m$, $\mathbf{x} \in X \subseteq \mathbb{R}^n$, $Z(\mathbf{y};\mathbf{x})$, $Z : Y \times X \to \mathbb{R}$, S denote choice (endogenous) variables, exogenous variables, loss function of the inventory system and feasible set, respectively [see Table 1 for notation].

The optimal policy $\mathbf{y}^*$ solves the problem

$$\mathcal{P}_1 = \left\{ \min_{\mathbf{y} \in S} Z(\mathbf{y};\mathbf{x}) \right. \tag{1}$$

Under the regularity condition $Z(\mathbf{y};\mathbf{x}) \in C^1(X)$, by Dini's implicit function theorem, the solution of $\mathcal{P}_1$ defines the differentiable function

$$\mathbf{y}^* = g(\mathbf{x}^*) : X \subseteq \mathbb{R}^n \to \mathbb{R}^m \tag{2}$$

Therefore, comparative statics can be applied. Borgonovo (2008) shows that the differential importance of exogenous variable x_i in respect of choice variable y_j (D_s^j) is given by

$$D_s^j(\mathbf{x}^*) = \left.\frac{\mathrm{d}_s y_j}{\mathrm{d} y_j}\right|_{\mathbf{x}^*} = \frac{\dfrac{\partial y_j(\mathbf{x}^*)}{\partial x_s}\mathrm{d}x_s}{\sum_{k=1}^n \dfrac{\partial y_j(\mathbf{x}^*)}{\partial x_k}\mathrm{d}x_k} \tag{3}$$

Table 1
Notation and symbols used throughout this work.

Symbol	Meaning
$\boldsymbol{\omega} = \{\omega_1, \omega_2, \ldots, \omega_s\}$	Vector of stochastic variable in the risk-coherent problems
$(\boldsymbol{\omega}, \otimes, F)$	Measure space
$\mathbf{y} = \{y_1, y_2, \ldots, y_m\}$	Endogenous variables (model output)
m	Number of endogenous variables
I	Number of Inventoried Items
Z	Loss function
$\rho(\cdot)$	Coherent risk-measure
$\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$	Exogenous variables (parameters)
n	Number of exogenous variables
g, $[\mathbf{y} = g(\mathbf{x})]$	Exogenous-endogenous variable relationship
$\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_Q\}$	Vector of parameter groups
Q	Number of parameter groups
$\mathbf{y}^*$	Optimal order policy
m	Number of choice variables
p	Size of the range partitions (number of scenarios)
$CVaR_\alpha^*$	CVaR at the point of optimum
$\mathbb{E}[Z^*]$	Expected loss at the optimum
$\xi^s_{i_1,i_2,\ldots,i_s}$	Group finite change sensitivity index of order s (FCSI)
$\varphi^k_{i_1,i_2,\ldots,i_k}$	Parameter finite change sensitivity index of order k for
ξ_i^T	Group total order FCSI
$a = \{a_1, a_2, \ldots, a_I\}$	Unit fixed costs per inventoried item
$r = \{r_1, r_2, \ldots, r_I\}$	Unit revenues per inventoried item
$c = \{c_1, c_2, \ldots, c_I\}$	Unit holding costs per inventoried item per unit time

where $\mathrm{d}_s y_j$ is the partial differential of y_j, $(\partial y_j / \partial x_s)$ is the partial derivative of y_j with respect to x_s and $\mathrm{d}x_s$ the infinitesimal change in x_s. $D_s^j(\mathbf{x}^*)$ generalizes comparative statics sensitivity measures (Borgonovo, 2008). If $Z(\mathbf{y};\mathbf{x})$ is twice continuously differentiable, by the fundamental theorem of comparative statics, $D_s^j(\mathbf{y}^*, \mathbf{x}^*)$ is expressed in terms the Hessian matrix of the Lagrangian function of $\mathcal{P}_1$ (Borgonovo, 2008). We now show that these regularity conditions are not satisfied in stochastic optimization problems with risk-coherent objective functions, generally.

In the presence of uncertainty, the economic consequences depend on $\mathbf{y}$, $\mathbf{x}$ and on the stochastic quantities ($\boldsymbol{\omega}$) involved in the problem Ruszczynski and Shapiro (2005), and Ahmed et al. (2007). For instance, in Grubbström (2008), Ahmed et al. (2007), Borgonovo and Peccati (2009a), and Gotoh and Takano (2007), ω is random demand. More in general, we let $\boldsymbol{\omega}$ be a random vector encompassing the stochastic variables of the problem and denote by $(\Omega, \mathcal{B}(\Omega), F)$ the probability space, with $\boldsymbol{\omega} \in \Omega, \subseteq \mathbb{R}^w$. One has $Z(\mathbf{y}, \boldsymbol{\omega}, \mathbf{x}) : X \times Y \times \Omega \to H \subseteq \mathbb{R}$. Consider the function $\rho = \rho(Z) : H \to \mathbb{R}$, with $\rho[Z] > 0$, $\forall Z \neq 0$. ρ is a coherent risk-measure, if it satisfies the translational invariance, positive homogeneity and subadditivity axioms of Artzner et al. (1999). Correspondingly, the optimal policy solves the problem (Ruszczynski and Shapiro, 2005):

$$\mathcal{P}_2 = \left\{ \min_{\mathbf{y} \in S(\mathbf{x})} \rho[Z(\mathbf{y}, \boldsymbol{\omega}, \mathbf{x})]. \right. \tag{4}$$

$\mathcal{P}_2$ is a non-linear stochastic program. Under suitable convexity conditions Ruszczynski and Shapiro (2005), $\mathcal{P}_2$ is feasible. We let $\mathbf{y}^*(\mathbf{x})$ represent a solution of $\mathcal{P}_2$.

We refer to Artzner et al. (1999) for a complete description of the axioms and implications of coherent measures of risk. In inventory management, coherent risk-measures have been applied for the first time in Ahmed et al. (2007). Gotoh and Takano (2007) generalize the problem presented in Ahmed et al. (2007) to a multi-item version. Borgonovo and Peccati (2009a) compare the optimal policies ($\mathbf{y}^*$) implied by different coherent risk-measures for the same inventory system. In Chen et al. (2007), a thorough analysis of risk aversion in inventory management is presented.