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Theoretical perspectives of trade-off analysis using DEA

Mette Asmild[a], Joseph C. Paradi[a,*], David N. Reese[b]

[a] Department of Chemical Engineering, The Centre for Management of Technology and Entrepreneurship, University of Toronto, 200 College Street, Toronto, Ont., Canada M5S 3E5
[b] TELUS, 11 King St West, FLR 14, Toronto, Ont., Canada M5H 4C7

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Abstract

Because of the piecewise linear nature of the frontier in data envelopment analysis (DEA), estimated marginal rates of substitution are only valid for infinitesimal, or small finite, changes in one or more variables. Analysing the impacts of very small changes is not adequate for many situations, where the consequences of substantial changes in variables are of interest to for instance managers. This paper develops methods for evaluating larger, i.e. non-marginal, tradeoffs between variables in DEA. The methods are capable of handling both scalar and additive changes. Finally, the approaches for evaluating basic pairwise tradeoffs are generalised in order to enable assessment of the impact on one or more throughputs of the change in one or more of the other throughputs.

Keywords: Data envelopment analysis (DEA); Rates of substitution; Marginal; Non-marginal; Trade-off

1. Introduction

Because of the inherent complexities in most production processes, it is usually not possible to change one decision parameter without affecting one or more other parameters. Such tradeoffs between inputs and/or outputs are important information for analysts and managers, as it enables evaluation of, or choices between, alternative locations in the production possibility space.

Tradeoffs between inputs and outputs, sometimes referred to as (marginal) rates of substitution or rates of return, represent partial derivatives or slopes of the efficient frontier. In data envelopment analysis (DEA) [1] ratios of the optimal multipliers provide this information, although have the inherent problem that the optimal multipliers may not be unique because of the piecewise linear nature of the DEA frontier. The result is that multiplier values cannot be used directly to study marginal rates without further considerations.

While there are papers on the subject of marginal rates of return at facet joints, past work mainly focuses on marginal changes within the facets themselves. Banker and Maindiratta [2] estimated technical inefficiencies and other production characteristics, such as rates of substitution and transformation. They did not, however, tackle the issues at the facet interfaces. Bessent et al. [3] developed a constrained facet analysis approach that provides a lower bound efficiency measure for organisational units that have a mix of resources or outputs that is different from any frontier point. This approach also yields marginal rates of productivity and substitution associated with the lower bound efficiency measure, but does not deal with the facet joints.

Charnes et al. [4], using data on the operations of Latin American airlines, developed an empirically efficient

* Corresponding author. Tel.: +1 416 978 6924x1; fax: +1 416 978 3877.
E-mail address: paradi@mie.utoronto.ca (J.C. Paradi).

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production function via a robustly efficient parametric frontier in a two-stage approach. The development used a multiplicative DEA model, where the marginal tradeoffs of the efficient production function are immediately available, without the problems of discontinuities of derivatives and numerical instabilities encountered at facet joints.

Clark [5] in a study of the US Air Force, developed a problem diagnoses methodology to help with the choices among alternative courses of action to improve the efficiency and effectiveness of combat units. In his work, he examined rates of substitution and marginal productivities in nearby frontier facets but barely considered the problems encountered at the hyperplane intersections. Olesen and Petersen [6] in their paper examined the problems arising from insufficient variation in data and the implication that some inputs/outputs can be substituted along the efficient frontier. They found that this is feasible if such substitution is made only in fixed proportions. They demonstrated that the existence of well-defined marginal rates of substitution along the estimated strongly efficient frontier segments requires the existence of full dimensional efficient facets (FDEFs). They also developed a test for the existence of the FDEFs.

Rosen et al. [7] directly address the problem of computing marginal rates on facets as well as at facet interfaces and present a general framework for the computation of tradeoffs in DEA, and for the application of the multiplier information.

However, marginal rates are limited to assessing the impact of infinitesimal changes of one or more variables on one or more other variables. As shown in [7], in the special case of DEA piecewise linear frontiers, finite differences methods, which utilise small, finite change, will also provide precise information on marginal rates. Analysing the impacts of these very small finite changes is, however, not adequate for many situations where the impacts of much larger changes are of interest, which is the motivation for the work presented here. In particular, for practical applications substantial (i.e. non-marginal) changes need to be considered to warrant any kind of effort or interest from managers.

Cooper et al. [8] propose an algorithm to search through the extreme points for cost improvements in the case of given prices or relative weights and constant outputs and thus consider non-marginal tradeoffs as well.

In this paper, we generalise the work of Rosen et al. [7] to enable analysis of non-marginal tradeoffs between variables, additive as well as scalar changes. Furthermore, we generalise the methods from pairwise tradeoffs to consider the impact of changes in one or more variables on one or more other variables. Some of the methods are illustrated in a simple empirical example. Our suggested approaches differ from those in [8] by not requiring prices or other pre-determined weights for the variables and by letting analysts or managers specify suggested values (absolute or relative) for changes in any subset of variables and calculate the consequences on any other subset of variables. This enables scenario analyses and answers to “what–if” questions.

The rest of this paper is structured as follows: Section 2 presents an adapted version of the approach in Rosen et al. [7] to evaluate basic additive, marginal pairwise tradeoffs. In Section 3 this approach is extended to deal with non-marginal tradeoffs, and to consider scalar changes as well as additive ones. In Section 4 these methods are generalised further to consider tradeoffs between more than two throughputs and Section 5 concludes the paper.

2. Marginal rates of substitution

Consider a set of n DMUs, $i=1,\dots,n$, each of which is characterised by a throughput vector $\mathbf{z}_i=\begin{pmatrix}-\mathbf{x}_i\\ \mathbf{y}_i\end{pmatrix}$, where the vector $\mathbf{x}_i$ contains m different inputs and the vector $\mathbf{y}_i$ contains s different outputs. Thus $\mathbf{z}_i\in\Re^{m+s}$ and the elements in $\mathbf{z}_i$ are also referred to without distinguishing between inputs and outputs as z_{ij}, where $j=1,\dots,(m+s)$. Denote the matrix of throughput vectors by $\mathbf{Z}$ where

$$\begin{aligned}\mathbf{Z}&=(\mathbf{z}_1,\dots,\mathbf{z}_i,\dots,\mathbf{z}_n)\\ &=\begin{pmatrix}\mathbf{z}_{11} & \cdots & \mathbf{z}_{i1} & \cdots & \mathbf{z}_{n1}\\ \vdots & & \vdots & & \vdots\\ \mathbf{z}_{1j} & \cdots & \mathbf{z}_{ij} & \cdots & \mathbf{z}_{nj}\\ \vdots & & \vdots & & \vdots\\ \mathbf{z}_{1(m+s)} & \cdots & \mathbf{z}_{i(m+s)} & \cdots & \mathbf{z}_{n(m+s)}\end{pmatrix}\\ &=\begin{pmatrix}-\mathbf{x}_1,\dots,-\mathbf{x}_i,\dots,-\mathbf{x}_n\\ \mathbf{y}_1,\dots,\mathbf{y}_i,\dots,\mathbf{y}_n\end{pmatrix}.\end{aligned}$$

Consider any throughput vector $\mathbf{z}\in\Re^{m+s}$ and let $F(\mathbf{z})$ be any distance function for which $F(\mathbf{z})=0$ if and only if $\mathbf{z}$ is on the production frontier and $F(\mathbf{z})<0$ if $\mathbf{z}$ is below the frontier, for instance the function $F(\mathbf{z})=\{\min\omega|[\mathbf{x},(1-\omega)\mathbf{y}]\in T\}$, where T is the production possibility set, i.e. all throughput vectors where $\mathbf{x}$ can produce $\mathbf{y}$. In a variable returns to scale DEA (see e.g. [9]) T is the piecewise linear convex envelopment of the set of observations, $T=\{\mathbf{z}|\mathbf{Z}\boldsymbol{\lambda}\geqslant\mathbf{z},\ \mathbf{1}^{\mathrm{T}}\boldsymbol{\lambda}=1\}$, where $\boldsymbol{\lambda}$ is a n-dimensional intensity vector and $\mathbf{1}$ a n-dimensional unity vector.

Let $\mathbf{z}_0$ be a point on the frontier. The marginal rate of throughput j to throughput k at $\mathbf{z}_0$, is given by

$$\mathrm{MRS}_{jk}(\mathbf{z}_0)=\left.\frac{\partial z_{0j}}{\partial z_{0k}}\right|_{\mathbf{z}_0}=-\left.\frac{(\partial F)/(\partial z_{0k})}{(\partial F)/(\partial z_{0j})}\right|_{\mathbf{z}_0} \tag{1}$$

and is the increase in throughput j that results when throughput k is increased by one unit, and all other throughputs are kept constant. Note that these derivatives only exist for points on the frontier and that we in this exposition only consider tradeoffs on the frontier. For non-frontier points one may first project the units to the frontier in whatever direction is desirable and then investigate the tradeoffs for the projected points.

These marginal rates of substitution can also be defined by the dual variables or optimal multipliers to the inputs and