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Bounds for present value functions with stochastic interest rates and stochastic volatility

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Abstract

The present value of a series of cash flows under stochastic interest rates has been investigated by many researchers. One of the main problems in this context is the fact that the calculation of exact analytical results for the distribution of this type of present values turns out to be rather complicated, and is known only for special cases. An interesting solution to this difficulty consists of determining computable upper bounds, as close as possible to the real distribution.

In the present contribution, we want to show how it is possible to compute such bounds for the present value of cash flows when not only the interest rates but also volatilities are stochastic. We derive results for the stop-loss premium and distribution of these bounds.

Keywords: Present value functions; Stochastic interest rates; Stochastic volatility

1. Introduction

Stochastic interest rates and their influence on present values of cash flow streams have been investigated by many researchers. Without claiming any completeness, for the last 10 years we can mention contributions of Beekman and Fuelling (1990, 1991), Yor (1993, 2001), Geman and Yor (1993), Sato and Yor (1998), Donati-Martin et al. (2000), Deelstra (1992), Parker (1994a,b,c, 1998), Dufresne (1989, 1990, 2001), Milevsky (1997, 1999), Milevsky and Posner (1998a,b), Artikis and Malliaris (1990), Artikis et al. (1993), Artikis and Voudouri (2000), Voudouri et al. (1999), Perry and Stadje (2000, 2001), next to the work of our group (De Schepper et al., 1992, 1994; De Schepper and Goovaerts, 1992; Vanneste et al., 1994, 1997; Vyncke et al., 2000). These contributions deal with the calculation of moments of such present values, with methods for deriving analytical results for the distribution of the present values, or with numerical approximations of these distributions.

When investigating present values, which are actually sums of dependent variables, one of the main problems that arise is the fact that due to the dependencies it is almost impossible to find the real distribution of such a sum. Therefore, exact results are known but in some special cases. In some recent papers (see Goovaerts et al., 2000; Goovaerts and Kaas, 2002; Kaas et al., 2000), we suggested to solve this problem by calculating upper bounds.

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Using the concept of comonotonicity, we are able to determine bounds in convexity order that are rather close to the original variable, and much easier to compute. For the meaning and consequences of this approach, we refer to Section 2.

One of the applications of this kind of problems is the investigation of the present value of a series of non-negative deterministic payments at times 1 up to n

$$A=\sum_{t=1}^{n}\alpha_t\,\mathrm{e}^{-Y_1-Y_2-\cdots-Y_t}, \tag{1}$$

where Y_t represents the stochastic continuous compounded rate of return over the period $[t-1,t]$ (see also Kaas et al., 2000).

In the classical assumption, prices are log-normally distributed, and thus the variables Y_t are independent and normally distributed. In other words,

$$Y_t\sim N(\mu_t,\sigma_t^2), \tag{2}$$

where μ_t and σ_t are constants.

In the present contribution, we will generalise this classical assumption by replacing the constant σ_t by a random variable $\tilde{\sigma}_t$, where we assume that the volatilities $\tilde{\sigma}_t$ for the periods $[t-1,t]$ are mutually independent variables. For any realisation σ_t, we then have that

$$Y_t|\tilde{\sigma}_t=\sigma_t\sim N(\mu_t,\sigma_t^2). \tag{3}$$

This idea has been borrowed from Taylor (1994).

In correspondence with the financial paradigma (see e.g. Gerber and Shiu, 1994), in Eq. (1) we should correct the variables Y_t by means of their volatility, or

$$A=\sum_{t=1}^{n}\alpha_t\,\mathrm{e}^{-(Y_1-1/2\tilde{\sigma}_1^2)-(Y_2-1/2\tilde{\sigma}_2^2)-\cdots-(Y_t-1/2\tilde{\sigma}_t^2)}, \tag{4}$$

$$A=\sum_{t=1}^{n}\alpha_t\,\mathrm{e}^{-Y(t)+1/2\Sigma(t)}, \tag{5}$$

where $Y(t)=Y_1+Y_2+\cdots+Y_t$ is used to denote the total compounded rate of return over the period $[0,t]$, and where $\Sigma(t)$ is defined as $\Sigma(t)=\tilde{\sigma}_1^2+\tilde{\sigma}_2^2+\cdots+\tilde{\sigma}_t^2$. The reason for this change by means of the volatility as suggested in Eqs. (4) and (5) has to be found in the fact that with this adaptation, for the (new) accumulated values we then have the identity

$$E[\mathrm{e}^{(Y_t-1/2\sigma_t^2)}]\,\mathrm{e}^{-\mu_t}=1. \tag{6}$$

Note that for the variable $Y(t)$ we have the obvious (conditional) moments

$$E[Y(t)|\tilde{\sigma}_1,\ldots,\tilde{\sigma}_t]=\mu_1+\cdots+\mu_t, \tag{7}$$

$$\mathrm{Var}[Y(t)|\tilde{\sigma}_1,\ldots,\tilde{\sigma}_t]=\tilde{\sigma}_1^2+\cdots+\tilde{\sigma}_t^2=\Sigma(t). \tag{8}$$

For the distributions of the variables $Y(t)$ and $\Sigma(t)$, we will use the notations $F_t(x)$ and $G_t(x)$, or

$$F_t(x)=\mathrm{Prob}[Y(t)\le x],\qquad f_t(x)=\frac{\mathrm{d}}{\mathrm{d}x}F_t(x), \tag{9}$$

and

$$G_t(x)=\mathrm{Prob}[\Sigma(t)\le x],\qquad g_t(x)=\frac{\mathrm{d}}{\mathrm{d}x}G_t(x). \tag{10}$$