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Household consumption, investment and life insurance

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ABSTRACT

This paper develops a continuous-time Markov model for utility optimization of households. The household optimizes expected future utility from consumption by controlling consumption, investments and purchase of life insurance for each person in the household. The optimal controls are investigated in the special case of a two-person household, and we present graphics illustrating how differences between the two persons affect the controls.

1. Introduction

Original consumption-investment problems are formulated in terms of optimizing utility of consumption and a terminal utility over a fixed time horizon for a single person; see Merton (1969, 1971). Richard (1975) included the problem of finding an optimal life insurance strategy, and formulated the problem of optimizing expected utility over an uncertain life time, where utility now arose from consumption and from leaving a positive amount of money upon death. Apart from introducing life insurance, Richard (1975) also modeled a continuous life time income, and found that the expected life time income had a positive effect on the demand for life insurance. Actually, the inclusion of an insurance decision in the personal finance optimization problem was first formulated in a discrete-time setting by Yaari (1965).

Since the path-breaking article of Hoem (1969), the continuous-time finite state Markov chain has played a prominent role in the theory of life insurance, and (Kraft and Steffensen, 2008) applied the continuous-time finite state Markov chain to the ideas established by Richard (1975). Kraft and Steffensen (2008) motivated the set-up by a personal finance model which allowed the customer to insure himself against disability, unemployment and similar personal risks.

Inspired by Kraft and Steffensen (2008) we use the Markov chain set-up for modeling household finance in the sense of optimizing expected future utility for a household consisting of economically and probabilistically dependent persons. The modeling is flexible enough to capture several interesting differences between the members of the household, and leads to closed form solutions for the optimal controls of investments, consumption of the household and purchase of life insurance for each of its members.

The paper is organized as follows. In Section 2, we present the general Markov model including the dynamics of the wealth of the household. Furthermore, we describe the assumptions concerning utility, and the general optimal value function for the problem. Section 3 presents the problem and the solution in the case of a one-person household, thereby setting the foundation for the multiple-person models. In Section 4, we solve the problem for a two-person household. We comment on the optimal control processes regarding consumption, investment and life insurance purchase, and in Section 5, we show numerical examples of these based on expectations to the investment market. In Section 6, we explain the mathematical induction techniques used for solving the multiple-person problem and write up the optimal controls in this case. Finally, in Section 7, we present ideas for further development of the model.

2. The general optimization problem

We let the state of the household be represented by a finite state multi-dimensional Markov chain, Z, and the state of the economy be represented by a standard Brownian motion W. These processes

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are assumed to be independent and defined on the measurable space $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is the natural filtration of (Z, W).

We let $\mathbb{P}$ and $\mathbb{P}^*$ be equivalent probability measures on the measurable space $(\Omega, \mathcal{F})$ and refer to $\mathbb{P}$ as the objective measure and $\mathbb{P}^*$ as the pricing measure, used for pricing both market risk (W) and life insurance risk (Z) by the insurance company. We hereby take the modern approach and consider life insurance policies as standard tradable financial contracts, as is done in e.g. Richard (1975) and Kraft and Steffensen (2008). Illiquidity issues could be dealt with on top of that, e.g. by introducing an illiquidity risk premium, but this is beyond the scope of this article.

It is essential for our studies below that the pricing measure exists such that pricing is unique and linear. Whereas this is conventional for e.g. equity risk, the assumption is a less conventional restriction for insurance risk. The pricing measure with respect to insurance risk may be equal to the objective measure with reference to diversification. Our results are not restricted to that case of zero market price on insurance risk but, as it can be seen below, the results become particularly simple in that case.

When modeling a household consisting of n persons, the state process Z takes values in $\{0, 1\}^n$, and by convention it starts in $\{0, 0, \ldots, 0\}$ at time 0. The n marginal processes of Z indicate, for each person, whether or not that person is dead, and thereby Z is given by

$$(Z_t)_{t\geq 0} = \left(Z_t^1, Z_t^2, \ldots, Z_t^n\right)_{t\geq 0},$$

where $Z^k = (Z_t^k)_{t\geq 0}$ counts the number of deaths for person k, $k \in \{1, 2, \ldots, n\}$.

The state process Z has jump intensities, $\hat{\mu}^{ij}$ under $\mathbb{P}$ and $\hat{\mu}^{*ij}$ under $\mathbb{P}^*$, and we denote the set of states to which Z can jump at time t by $\mathcal{Z}_t$. As we do not allow for multiple deaths in a small time interval or for resurrection, the number of states in $\mathcal{Z}_t$ equals the number of persons being alive at time t.

For any given $i = (i_1, i_2, \ldots, i_n)$ and $j = (j_1, j_2, \ldots, j_n)$, we write the transition rate functions

$$\hat{\mu}_t^{ij} = \prod_{l=1}^{n} (1 - Z_t^l)^{1-i_l} (Z_t^l)^{i_l} \mu_t^{ij},$$

$$\hat{\mu}_t^{*ij} = \prod_{l=1}^{n} (1 - Z_t^l)^{1-i_l} (Z_t^l)^{i_l} \mu_t^{*ij},$$

for some deterministic continuous transition rate functions μ_t^{ij} and μ_t^{*ij}. These functions are non-null only for i and j such that the transition $i \to j$ is possible, i.e. $i_k = 0$ and $j_k = 1$ for exactly one k and $i_l = j_l$ for $l \neq k$. In order to have well-defined problems we assume that $\mu_t^{ij} \to \infty$ and $\mu_t^{*ij} \to \infty$ for those pairs of states (i, j) for which the transition $i \to j$ is possible. That implies in particular that

$$\lim_{t\to\infty} \mathbb{P}(Z_t = \{1, 1, \ldots, 1\}) = \lim_{t\to\infty} \mathbb{P}^*(Z_t = \{1, 1, \ldots, 1\}) = 1.$$

The compensated jumping process is a martingale under the respective measures, meaning that $M = Z - \int \hat{\mu}$ is a martingale under $\mathbb{P}$ and $M^* = Z - \int \hat{\mu}^*$ is a martingale under $\mathbb{P}^*$. In particular, we will use the marginal processes, and for $j \in \mathcal{Z}_t$ write

$$\mathrm{d}M_t^{*Z_t j} = dZ_t^{\psi(Z_t, j)} - \mu_t^{*Z_t j}\mathrm{d}t$$

for the dynamics at time t of the marginal martingale given Z_t, where $\psi(i, j)$ gives the coordinate of Z that changes from 0 to 1 upon a jump of Z from state i to state j. Note that $\hat{\mu}^{*Z_t j} = \mu^{*Z_t j}$ since $j \in \mathcal{Z}_t$.

Wealth dynamics

The household decides on an optimal allocation of wealth in a risky asset and a risk-free asset at all times. The household has access to an investment market consisting of a bond (B) and a stock (S) with Black–Scholes dynamics

$$\mathrm{d}B_t = rB_t\mathrm{d}t,$$
$$B_0 = b_0 > 0,$$
$$\mathrm{d}S_t = \alpha S_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t,$$
$$S_0 = s_0 > 0,$$

where r, α and σ are constants and W is a standard Brownian motion. The proportion of household wealth invested in the stock is described by the process π. This simple investment market model is chosen since the focus here is on the life insurance decisions, but the results can be generalized to more advanced investment market models.

Allowing the household to purchase life insurance for each person at all times, the household wealth, X, follows the dynamics

$$\mathrm{d}X_t = (r + \pi_t(\alpha - r))X_t\mathrm{d}t + \pi_t\sigma X_t\mathrm{d}W_t + a_t^{Z_t}\mathrm{d}t - c_t\mathrm{d}t + \sum_{j\in\mathcal{Z}_{t-}} S_t^j\mathrm{d}M_t^{*Z_{t-}j},$$

$$X_0 = x_0,$$

where a^j is the deterministic income process corresponding to state j, $j \in \{0, 1\}^n$, and c is the total consumption process of the household. The processes S^j are the sums insured such that S^j is the amount paid out upon a jump of Z from state Z_{t-} to state j, and $\mu_t^{*Z_{t-}j} S_t^j$ is the natural premium intensity that the household pays at time t for that life insurance. The linearity of the premium as a function of the sum insured is a consequence of assuming existence of a pricing measure. This linearity is essential for our studies and the application of our results below is restricted to that situation. The special case of zero market price of insurance risk corresponds to setting $\mu^* = \mu$ and represents a relevant and particularly simple special case.

In practical, building of reserves in insurance companies is needed for trading life annuities. That could be dealt with by formulating our optimization problem with two types of wealth, personal wealth and institutional wealth, as it is done by Kraft and Steffensen (2008). However, they find that if we impose no constraints on these wealth processes and allow utility to depend only on the sum of them, then we need not model two separate wealth processes (the optimization problem with two wealth processes results in the same optimal controls as the problem with only one). Since we allow for utility of consumption only, in our case it is sufficient to model one wealth process of the household.

A life annuitant leaves his institutional wealth to the insurance company upon death. When we do not distinguish between personal and institutional wealth, this is just seen as a sum paid out of the wealth of the individual to the insurance company. Therefore, we speak of a negative sum insured as a life annuity payment. Thus, not restricting the sum insured to be positive is essentially equivalent to allowing for purchase of life annuities.

The optimization problem

We consider the problem of maximizing expected utility for the household, where the utility is assumed to come from consumption only. In particular, we assume that there is no utility from leaving a positive amount of money at the time when the last person in the household dies (as is done by e.g. Richard (1975) and Kraft and Steffensen (2008)). Writing $u^j(t, c)$ for the utility of consuming c at time t, given that $Z_t = j$, the optimization problem is

$$\sup_{q\in\mathcal{Q}[0,\infty)} \mathbb{E}_{0,x,0}\left(\int_0^\infty \sum_{j\in\{0,1\}^n} \mathbb{1}_{\{Z_s=j\}} u^j(s, c_s)\mathrm{d}s\right),$$