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Fuzzy weighted average: The linear programming approach via Charnes and Cooper's rule

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Abstract

To improve the method of Dong and Wong (Fuzzy Sets and Systems 21 (1987) 183–199) for obtaining the fuzzy weighted average, a steepest descent/ascent method was proposed by Liou and Wang (Fuzzy Sets and Systems 49 (1992) 307–315). In this paper, we propose to replace the steepest descent/ascent method by linear programming, which is a much more powerful approach for large problems. The reduction of the original nonlinear programming problem into a linear one is achieved by the use of the Charnes and Cooper's linear transformation method. The example used by Dong and Wong is again used to illustrate the approach.

Keywords: Multiple criteria evaluation; Fuzzy weighted average; Extension principle; Linear transformation

1. Introduction

In multiple criteria decision making, the environment is frequently vague and difficult to define. Thus, the weighted average is frequently expressed in fuzzy numbers as follows:

$$y = f(x_1, x_2, \ldots, x_n, w_1, w_2, \ldots w_n) = \frac{w_1x_1 + w_2x_2 + \cdots + w_nx_n}{w_1 + w_2 + \cdots + w_n}, \quad (1)$$

where $x_1, x_2, \ldots x_n$ are fuzzy numbers in fuzzy sets $A_1, A_2, \ldots, A_n$; $w_1, w_2, \ldots, w_n$ are fuzzy weights in fuzzy sets $W_1, W_2, \ldots, W_n$, and y is the dependent or output fuzzy variable in fuzzy set B. To aggregate this fuzzy expression so that the value of y can be obtained, fuzzy operations are needed. In principle, these operations can be carried out by the use of Zadeh's extension principle or fuzzy extended operations. However, the implementation of this solution procedure is complicated due to the fact that the resulting problem by applying the extension principle is a nonlinear programming problem and, furthermore, the conjunction and disjunction operations can lead to irregular membership functions. Dong and Wong [3] proposed a computational algorithm based on α-cuts and interval analysis to overcome this difficulty. Their method provides a discrete but exact solution to the fuzzy weighted average. Later, Liou and Wang [4] improved this approach by using a steepest descent/ascent method to obtain the optima in Steps 3 and 4 of the Dong and Wong's procedure. Based on Dong and Wong's procedure, 2^{2n} evaluations were needed for each *α-cut* or interval, Liou and Wang reduced this requirement to 2^{n+1}.

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However, even with the above improvement, the computational requirements are still very high even for moderate sized problems. In this paper, we propose to replace the steepest descent/ascent optimization procedure by linear programming, which is, of course, a much more powerful technique than the steepest descent/ascent approach. The original nonlinear fractional programming problem is transformed into a linear programming problem by the use of the Charnes and Cooper's linear transformation [1,2]. The advantage of this approach is that the reduction to linear problem is very straightforward and simple. The original example used by Dong and Wong is again used to illustrate the approach.

2. Dong and Wong's procedure [3]

Dong and Wong's approach is based on α-cuts to obtain the different intervals and combinatorial interval analysis to avoid some of the difficulties such as the multiple occurrence of variables. For later reference, their procedure is briefly summarized in the following:

1. Discretize the complete range of the membership $[0,1]$ of the fuzzy number into the following finite number of m-values or m α-cuts: $\alpha_1, \alpha_2, \ldots, \alpha_m$, where the degree of accuracy depends on the number of α-cuts or the number m.
2. For each α_j, find the corresponding intervals for A_i in x_i and W_i in w_i. Denote the end points of the intervals of x_i and w_i by $[a_i, b_i]$ and $[c_i, d_i]$, $i = 1,2,\ldots,n$, respectively.
3. Construct the 2^{2n} distinct permutations of the $2n$ array $(x_1, x_2, \ldots, x_n;\ w_1, w_2, \ldots, w_n)$.
4. Compute $y_k = f(x_{k1}, x_{k2}, \ldots, x_{kn};\ w_{k1}, w_{k2}, \ldots, w_{kn})$, where $(x_{k1}, x_{k2}, \ldots, x_{kn};\ w_{k1}, w_{k2}, \ldots, w_{kn})$, is the kth permutation of the 2^{2n} distinct permutations, $k = 1,2,\ldots,2^{2n}$. Then the desired intervals for y is

$$y = \left[\min_k y_k, \max_k y_k\right]. \tag{2}$$

5. Repeat Steps 2–4 for every α_j, $j = 1,2,\ldots,m$.

Since there are 2^{2n} permutations, there are 2^{2n} evaluations for each α-cut.

3. Liou and Wang's improvement [4]

Liou and Wang introduced the following notations:

$$f_L(w_1, w_2, \ldots, w_n) = \frac{w_1a_1 + w_2a_2 + \cdots + w_na_n}{w_1 + w_2 + \cdots + w_n}, \tag{3}$$

$$f_U(w_1, w_2, \ldots, w_n) = \frac{w_1b_1 + w_2b_2 + \cdots + w_nb_n}{w_1 + w_2 + \cdots + w_n} \tag{4}$$

and proved the following theorems:

$$\begin{aligned}\text{min: } & f(x_1, x_2, \ldots, x_n, w_1, w_2, \ldots, w_n) \\ & = \text{min: } f_L(w_1, w_2, \ldots, w_n),\end{aligned} \tag{5}$$

$$\begin{aligned}\text{max: } & f(x_1, x_2, \ldots, x_n, w_1, w_2, \ldots, w_n) \\ & = \text{max: } f_U(w_1, w_2, \ldots, w_n).\end{aligned} \tag{6}$$

Because of the above theorems, the weight vector w_i, $i = 1,2,\ldots,n$ are the only unknowns in the original optimization problem of Liou and Wong, and the unknowns for the other vector x_i, $i = 1,2,\ldots,n$, are obtained by the use of the above theorems. In other words, for every α_j, the left and right end points of the interval of y can be obtained by the evaluations of the end points of $(w_1, w_2, \ldots, w_n)$ only. Remember that the left and right end points for the intervals of w_i, are c_i and d_i, respectively, with $i = 1,2,\ldots,n$. Thus, the number of evaluations for each α_j is reduced from 2^{2n} to 2^{n+1}. These investigators also proved three more theorems, by the use of which, they developed a procedure to obtain the optimum based on the steepest descent/ascent approach.

4. Charnes and Cooper's transformation [1,2]

Instead of using the steepest descent/ascent approach to solve the nonlinear fractional programming problem, we wish to use the linear programming approach by first transforming this nonlinear problem into a linear one by using the Charnes and Cooper's linear transformation technique. From the two theorems represented by Eqs. (5) and (6), the minimum and maximum for the fuzzy weighted average for each given α_j can be obtained by solving the