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Optimality conditions for linear programming problems with fuzzy coefficients

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Abstract

The optimality conditions for linear programming problems with fuzzy coefficients are derived in this paper. Two solution concepts are proposed by considering the orderings on the set of all fuzzy numbers. The solution concepts proposed in this paper will follow from the similar solution concept, called the nondominated solution, in the multiobjective programming problem. Under these settings, the optimality conditions will be naturally elicited.

Keywords: Fuzzy numbers; Nondominated solutions; (crisp) Fuzzy constraints

1. Introduction

The occurrence of fuzziness in the real world is inevitable owing to some unexpected situations. Therefore, imposing fuzziness upon conventional optimization problems becomes an interesting research topic. The collection of papers on fuzzy optimization edited by Słowiński [1] and Delgado et al. [2] gives the main stream of this topic. Lai and Hwang [3,4] also give an insightful survey. On the other hand, the book edited by Słowiński and Teghem [5] provides comparisons between fuzzy optimization and stochastic optimization for multiobjective programming problems.

Bellman and Zadeh [6] inspired the development of fuzzy optimization by providing the aggregation operators, which combined the fuzzy goals and fuzzy decision space. After this motivation and inspiration, there appeared a lot of articles dealing with fuzzy optimization problems. Some interesting articles are Buckley [7,8], Julien [9] and Luhandjula et al. [10] using possibility distribution, Herrera et al. [11] and Zimmermann [12,13] using fuzzified constraints and objective functions, Inuiguchi et al. [14,15] using modality measures, Tanaka and Asai [16] using fuzzy parameters, and Lee and Li [17–19] considering the de Novo programming problem.

The duality of the fuzzy linear programming problem was firstly studied by Rodder and Zimmermann [20] considering the economic interpretation of the dual variables. After that, many interesting results regarding the duality of the fuzzy linear programming problem was investigated by Bector et al. [21–23], Liu et al. [24], Ramík [25], Verdegay [26] and Wu [27]. In this paper, we investigate the optimality conditions for linear programming problems with fuzzy coefficients.

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In Section 2, we introduce some basic properties and arithmetics of fuzzy numbers. In Section 3, we formulate two linear programming problems with fuzzy coefficients. One considers crisp (conventional) linear constraints, and the other considers fuzzy linear constraints. Two solution concepts are proposed for these two problems. In Section 4, we derive the optimality conditions for these two problems by introducing the multipliers. Finally, in Section 5, three examples are provided to illustrate the discussions in linear programming problems with fuzzy coefficients.

2. Arithmetics of fuzzy numbers

Let $\mathbb{R}$ be the set of all real numbers. The fuzzy subset $\tilde{a}$ of $\mathbb{R}$ is defined by a function $\xi_{\tilde{a}} : \mathbb{R} \to [0, 1]$, which is called a *membership function*. The α-level set of $\tilde{a}$, denoted by $\tilde{a}_\alpha$, is defined by $\tilde{a}_\alpha = \{x \in \mathbb{R} : \xi_{\tilde{a}}(x) \geq \alpha\}$ for all $\alpha \in (0, 1]$. The 0-level set $\tilde{a}_0$ is defined as the closure of the set $\{x \in \mathbb{R} : \xi_{\tilde{a}}(x) > 0\}$, i.e., $\tilde{a}_0 = \text{cl}(\{x \in \mathbb{R} : \xi_{\tilde{a}}(x) > 0\})$.

Definition 2.1. We denote by $\mathcal{F}(\mathbb{R})$ the set of all fuzzy subsets $\tilde{a}$ of $\mathbb{R}$ with membership function $\xi_{\tilde{a}}$ satisfying the following conditions:

(i) $\tilde{a}$ is normal, i.e., there exists an $x \in \mathbb{R}$ such that $\xi_{\tilde{a}}(x) = 1$;
(ii) $\xi_{\tilde{a}}$ is quasi-concave, i.e., $\xi_{\tilde{a}}(\lambda x + (1 - \lambda)y) \geq \min\{\xi_{\tilde{a}}(x), \xi_{\tilde{a}}(y)\}$ for all $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$;
(iii) $\xi_{\tilde{a}}$ is upper semicontinuous, i.e., $\{x \in \mathbb{R} : \xi_{\tilde{a}}(x) \geq \alpha\} = \tilde{a}_\alpha$ is a closed subset of U for each $\alpha \in (0, 1]$;
(iv) the 0-level set $\tilde{a}_0$ is a compact subset of $\mathbb{R}$.

The member $\tilde{a}$ in $\mathcal{F}(\mathbb{R})$ is called a *fuzzy number*.

Suppose now that $\tilde{a} \in \mathcal{F}(\mathbb{R})$. From Zadeh [28], the α-level set $\tilde{a}_\alpha$ of $\tilde{a}$ is a convex subset of $\mathbb{R}$ for each $\alpha \in [0, 1]$ from condition (ii). Combining this fact with conditions (iii) and (iv), the α-level set $\tilde{a}_\alpha$ of $\tilde{a}$ is a compact and convex subset of $\mathbb{R}$ for each $\alpha \in [0, 1]$, i.e., $\tilde{a}_\alpha$ is a closed interval in $\mathbb{R}$ for each $\alpha \in [0, 1]$. Therefore, we also write $\tilde{a}_\alpha = \left[\tilde{a}_\alpha^L, \tilde{a}_\alpha^U\right]$.

Definition 2.2. Let $\tilde{a}$ be a fuzzy number. We say that $\tilde{a}$ is *nonnegative* if $\tilde{a}_\alpha^L \geq 0$ for all $\alpha \in [0, 1]$. We say that $\tilde{a}$ is *positive* if $\tilde{a}_\alpha^L > 0$ for all $\alpha \in [0, 1]$. We say that $\tilde{a}$ is *nonpositive* if $\tilde{a}_\alpha^U \leq 0$ for all $\alpha \in [0, 1]$. We say that $\tilde{a}$ is *negative* if $\tilde{a}_\alpha^U < 0$ for all $\alpha \in [0, 1]$.

Remark 2.1. Let $\tilde{a}$ be a fuzzy number. Then $\tilde{a}_\alpha^L \leq \tilde{a}_\alpha^U$ for all $\alpha \in [0, 1]$. Therefore if $\tilde{a}$ is nonnegative then $\tilde{a}_\alpha^L \geq 0$ and $\tilde{a}_\alpha^U \geq 0$ for all $\alpha \in [0, 1]$, and if $\tilde{a}$ is positive then $\tilde{a}_\alpha^L > 0$ and $\tilde{a}_\alpha^U > 0$ for all $\alpha \in [0, 1]$. We can have similar consequences for nonpositive and negative fuzzy numbers.

Let "$\odot$" be any binary operations $\oplus$ or $\otimes$ between two fuzzy numbers $\tilde{a}$ and $\tilde{b}$. The membership function of $\tilde{a} \odot \tilde{b}$ is defined by

$$\xi_{\tilde{a}\odot\tilde{b}}(z) = \sup_{x \circ y = z} \min\{\xi_{\tilde{a}}(x), \xi_{\tilde{b}}(y)\}$$

using the extension principle in Zadeh [29], where the operations $\odot = \oplus$ and $\otimes$ correspond to the operations $\circ = +$ and $\times$, respectively. Then we have the following results.

Proposition 2.1. *Let $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R})$. Then we have*

(i) $\tilde{a} \oplus \tilde{b} \in \mathcal{F}(\mathbb{R})$ *and*

$$(\tilde{a} \oplus \tilde{b})_\alpha = \left[\tilde{a}_\alpha^L + \tilde{b}_\alpha^L, \tilde{a}_\alpha^U + \tilde{b}_\alpha^U\right];$$

(ii) $\tilde{a} \otimes \tilde{b} \in \mathcal{F}(\mathbb{R})$ *and*

$$(\tilde{a} \otimes \tilde{b})_\alpha = \left[\min\left\{\tilde{a}_\alpha^L\tilde{b}_\alpha^L, \tilde{a}_\alpha^L\tilde{b}_\alpha^U, \tilde{a}_\alpha^U\tilde{b}_\alpha^L, \tilde{a}_\alpha^U\tilde{b}_\alpha^U\right\}, \max\left\{\tilde{a}_\alpha^L\tilde{b}_\alpha^L, \tilde{a}_\alpha^L\tilde{b}_\alpha^U, \tilde{a}_\alpha^U\tilde{b}_\alpha^L, \tilde{a}_\alpha^U\tilde{b}_\alpha^U\right\}\right].$$

The following proposition is very useful for discussing the optimality conditions.

Proposition 2.2. *Let $\tilde{a}$ be a nonnegative fuzzy number and $\tilde{b}$ be a nonpositive fuzzy number. If $\tilde{a} \otimes \tilde{b} = \tilde{0}$, then $\tilde{a}_\alpha^L\tilde{b}_\alpha^L = \tilde{a}_\alpha^L\tilde{b}_\alpha^U = \tilde{a}_\alpha^U\tilde{b}_\alpha^L = \tilde{a}_\alpha^U\tilde{b}_\alpha^U = 0$ for all $\alpha \in [0, 1]$.*